

Localization Lengths and Boltzmann Limit for the Anderson Model at Small Disorders in Dimension 3

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We prove lower bounds on the localization length of eigenfunctions in the three-dimensional Anderson model at weak disorders. Our results are similar to those obtained by Schlag, Shubin and Wolff, [*J. Anal. Math.* 88 (2002)], for dimensions one and two. We prove that with probability one, most eigenfunctions have localization lengths bounded from below by $O\left(\frac{\lambda^{-2}}{\log \frac{1}{\lambda}}\right)$, where λ is the disorder strength. This is achieved by time-dependent methods which generalize those developed by Erdős and Yau [*Commun. Pure Appl. Math.* **LIII**: 667–753 (2003)] to the lattice and non-Gaussian case. In addition, we show that the macroscopic limit of the corresponding lattice random Schrödinger dynamics is governed by a linear Boltzmann equation.

KEY WORDS: Anderson model; hydrodynamic limits; localization lengths; random Schrödinger operators; weak coupling limit; quantum kinetic theory.

1. INTRODUCTION

The Anderson model in dimension d is defined by the discrete random Schrödinger operator

$$(H_\omega \psi)(x) = -\frac{1}{2}(\Delta \psi)(x) + \lambda \omega(x) \psi(x),$$

acting on $\ell^2(\mathbb{Z}^d)$, where λ is a small coupling constant, accounting for the strength of the disorder.

$$(\Delta \psi)(x) := 2d\psi(x) - \sum_{|x-y|=1} \psi(y)$$

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is the nearest neighbor lattice Laplacian, and $\omega(x)$ shall, for $x \in \mathbb{Z}^d$, be bounded, i.i.d. random variables. In the present paper, we study the case $d = 3$, and prove that with probability one, most eigenfunctions of H_ω have localization lengths bounded from below by $O\left(\frac{\lambda^{-2}}{\log \frac{1}{\lambda}}\right)$. In contrast to $d = 1, 2$, we note that there are no restrictions on the energy range for the validity of this result. Furthermore, we derive the macroscopic limit of the quantum dynamics in this system, and prove that it is governed by the linear Boltzmann equations.

The present paper is closely related to work of Erdős and Yau,⁽³⁾ in which the weak coupling and hydrodynamic limit has been derived for a random Schrödinger equation in the continuum \mathbb{R}^d , $d = 2, 3$, for a Gaussian random potential. For macroscopic time and space variables (T, X) , microscopic variables (t, x) , and the scaling $(X, T) = \lambda^2(x, t)$, where λ is the coupling constant in the continuum analogue of H_ω , these authors established in the limit $\lambda \rightarrow 0$ that the macroscopic dynamics is governed by a *linear Boltzmann equation*, and thus ballistic, globally in $T > 0$. We note that the corresponding local in $T > 0$ result was first proved by Spohn.⁽⁹⁾ For a time scale larger than $O(\lambda^{-2})$, Erdős, *et al.* have very recently succeeded in establishing that the macroscopic dynamics in $d = 3$ is determined by a diffusion equation.⁽⁴⁾

The problem addressed in the present paper is, on the other hand, closely related to recent work of Schlag *et al.*⁽⁸⁾ Based on techniques of harmonic analysis, it was established in ref. 8 for the Anderson model at small disorders in $d = 1, 2$ that with probability one, most eigenstates are in frequency space concentrated on shells of thickness $\leq \lambda^2$ in $d = 1$, and $\leq \lambda^{2-\delta}$ in $d = 2$. The eigenenergies are required to be bounded away from the edges of the spectrum of $-\frac{1}{2}\Delta_{\mathbb{Z}^d}$, and in $d = 2$, also away from its center. By the uncertainty principle, this implies lower bounds of order $O(\lambda^{-2})$ in $d = 1$, and $O(\lambda^{-2+\delta})$ in $d = 2$, on the localization lengths in position space. Closely related to their work are the papers by Magnen *et al.*^(5,6) and by Poirot⁽⁷⁾, which address properties of the Greens functions associated to H_ω .

The proof the main results in the present paper uses an extension of the time-dependent techniques of Erdős and Yau⁽³⁾ to the lattice, and to non-Gaussian random potentials. Higher correlations, which are now abundant, are shown to have an insignificant effect, hence the character of our results does not differ from that obtained in the Gaussian case. Furthermore, bounds on the amplitudes of certain Feynman diagrams of “crossing” structure are much harder to obtain in the lattice than in the continuum model, due to the significantly more complicated

geometry of energy level surfaces. We have adapted part of our notation and nomenclature to ref. 3, in order to facilitate the referencing of results.

The link between the lower bounds on the localization lengths of eigenfunctions, and the Schrödinger dynamics generated by H_ω is a joint result with Erdős and Yau included in this paper. The author is deeply grateful to them for their support and generosity.

2. DEFINITION OF THE MODEL AND STATEMENT OF THE MAIN THEOREM

We consider the discrete random Schrödinger operator

$$H_\omega = -\frac{1}{2}\Delta + \lambda V_\omega \tag{1}$$

acting on $\psi \in \ell^2(\mathbb{Z}^3)$. The impurity potential is given by

$$V_\omega(x) = \sum_{y \in \mathbb{Z}^3} \omega_y \delta(x - y), \tag{2}$$

where ω_y are bounded, independent, identically distributed random variables, of mean 0, and normalized variance. For each $x \in \mathbb{Z}^3$, ω_x is a random variable on a single site probability space (J, F, μ) , where J is a Borel subset of \mathbb{R} with $|J| := \sup_{\omega, \omega' \in J} |\omega - \omega'| < \infty$, F is the σ -algebra of Borel subsets of J , and μ is a probability measure on F . V_ω is a random field over \mathbb{Z}^3 realized on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = \times_{\mathbb{Z}^3} J$, where \mathcal{F} is the σ -algebra generated by the cylinder sets induced by F , and the probability measure \mathbb{P} is given by $\times_{\mathbb{Z}^3} \mu$. For simplicity, we assume μ to be even, $\mu(I) = \mu(-I)$, for all $I \in F$. Then, $\mathbb{E}[\omega_x^{2m+1}] = 0 \forall x \in \mathbb{Z}^3, \forall m \geq 0$. This reduces some of the notation in our analysis, but for our methods to apply, only $\mathbb{E}[\omega_x] = 0$ is necessary. Clearly, $\mathbb{E}[\omega_x^{2m}] < |J|^{2m}$ for all m , but we shall here use the moment bounds

$$\mathbb{E}[\omega_x^{2m}] =: \tilde{c}_{2m} \leq (2m)! c_V, \quad \tilde{c}_2 = 1, \quad \forall x \in \mathbb{Z}^3, \quad \forall m \geq 1, \tag{3}$$

for a constant $c_V < \infty$ which is independent of m and $|J|$. This allows for a generalization of our results to cases of unbounded random variables, which we expect to be straightforward. We shall here not further discuss the latter issue.

We use the convention

$$\begin{aligned} \hat{f}(k) &\equiv \mathcal{F}(f)(k) = \sum_{x \in \mathbb{Z}^3} e^{-2\pi i k \cdot x} f(x), \\ \check{g}(x) &\equiv \mathcal{F}^{-1}(g)(x) = \int_{\mathbb{T}^3} dk g(k) e^{2\pi i k \cdot x} \end{aligned} \tag{4}$$

for the Fourier transform and its inverse. Then,

$$\begin{aligned} (\Delta f)^\wedge(k) &= -2 e_{\Delta}(k) \hat{f}(k), \\ e_{\Delta}(k) &:= \sum_{i=1}^3 (1 - \cos(2\pi k_i)) = 2 \sum_{i=1}^3 \sin^2(\pi k_i) \end{aligned} \tag{5}$$

is the expression for the kinetic energy in frequency space.

Let $L \gg \lambda^{-2}$, and $\Lambda_L := [-L, L]^3 \cap \mathbb{Z}^3$. For $m \in \mathbb{N}_0$ and $\ell \in \mathbb{R}$ with $m \leq \ell \ll L$, let

$$\begin{aligned} h_{\ell}(m) &:= \begin{cases} 1 & \text{if } 0 \leq m \leq \frac{\lfloor \ell \rfloor}{2} \\ 2 - \frac{2m}{\lfloor \ell \rfloor} & \text{if } \frac{\lfloor \ell \rfloor}{2} < m \leq \lfloor \ell \rfloor \\ 0 & \text{otherwise} \end{cases} \\ K_{\ell}(x) &:= \prod_{j=1}^3 h_{\ell}(|x_j|) \\ R_{x,\delta,\ell}(y) &:= K_{\ell}(x - y) - K_{\delta\ell}(x - y). \end{aligned} \tag{6}$$

We remark that \hat{K}_{ℓ} is a product of differences of Fejér kernels, and that for $x \in \Lambda_L$ and $\delta > 0$, $R_{x,\delta,\ell}(y)$ is an approximate characteristic function supported on a cubical shell of side length 2ℓ centered at x , and thickness $(1 - \delta)\ell$.

The author thanks Yau and Erdős for the following observation, which is the key to linking the localization length of eigenvectors to the dynamics generated by H_{ω} . For a fixed realization of the random potential, let $\{\psi_{\alpha}^{(L)}\}$ denote an orthonormal basis in $\ell^2(\Lambda_L)$ of eigenfunctions of H_{ω} restricted to Λ_L ,

$$\begin{aligned} (H_{\omega} - e_{\alpha}^{(L)})\psi_{\alpha}^{(L)} &= 0 \quad \text{on } \Lambda_L \quad \text{and} \\ \psi_{\alpha}^{(L)} &= 0 \quad \text{on } \partial\Lambda_L := \Lambda_{L+1} \setminus \Lambda_L \end{aligned} \tag{7}$$

for

$$\begin{aligned} \alpha \in \mathcal{A}_L &:= \{1, \dots, |\Lambda_L|\} \\ e_{\alpha}^{(L)} &\in \mathbb{R}. \end{aligned} \tag{8}$$

For ε small, let

$$\mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)} := \left\{ \alpha \mid \sum_{x \in \Lambda_L} |\psi_\alpha^{(L)}(x)| \|R_{x,\delta,\ell} \psi_\alpha^{(L)}\|_{\ell^2(\Lambda_L)} < \varepsilon \right\} \subset \mathcal{A}_L. \tag{9}$$

Then, $\{\psi_\alpha^{(L)}\}_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}}$ contains the class of exponentially localized eigenstates concentrated in balls of radius $O(\frac{\delta \ell}{\log \ell})$ or smaller, where we emphasize that δ is independent of ℓ . The additional factor $\log \ell$ in the denominator compensates a volume factor $O(\ell^{3/2})$, which arises due to the fact that $|\psi_\alpha^{(L)}(x)|$ appears only linearly, and not quadratically in the sum. Our main result is the following theorem.

Theorem 2.1. Assume $L \gg \lambda^{-2}$, and that $\{\psi_\alpha^{(L)}\}$ is an orthonormal H_ω -eigenbasis in $\ell^2(\Lambda_L)$, satisfying (7) with $\alpha \in \mathcal{A}_L$, and $e_\alpha^{(L)} \in \mathbb{R}$. Then, for $\lambda^{\frac{14}{15}} < \delta < 1$ and $\varepsilon_\delta := \delta^{\frac{3}{7}}$,

$$\mathbb{E} \left[\frac{|\mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon_\delta,\delta,\lambda^{-2}}^{(\omega)}|}{|\mathcal{A}_L|} \right] \geq 1 - C\delta^{\frac{3}{14}} - C\lambda^{-2}L^{-1},$$

for finite constants C that are uniform in L, δ, λ . Furthermore,

$$\mathbb{P} \left[\liminf_{L \rightarrow \infty} \frac{|\mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon_\delta,\delta,\lambda^{-2}}^{(\omega)}|}{|\mathcal{A}_L|} \geq 1 - C\delta^{\frac{3}{14}} \right] = 1$$

for $\lambda > 0$ sufficiently small, and a finite constant C that is uniform in λ and δ .

We note that in contrast to the results for dimension $d = 1, 2$ established in ref. 8, there is no restriction in dimension 3 on the range of values of $e_\alpha^{(L)}$. Furthermore, we emphasize that the correction to the lower bound of order $O(\lambda^{-2})$ on the localization length is only logarithmic, while the bound obtained in ref. 8 for $d = 2$ is of order $O(\lambda^{-2+\varepsilon})$, for any $\varepsilon > 0$.

3. PROOF OF THE MAIN THEOREM

Key to Theorem 2.1 is the following lemma, which establishes a link between the localization length of eigenvectors of H_ω and the dynamics generated by H_ω .

Lemma 3.1 (*Joint with L. Erdős and H.-T. Yau*). Let $\{\psi_\alpha^{(L)}\}$ denote an orthonormal basis in $\ell^2(\Lambda_L)$, consisting of eigenvectors of H_ω satisfying (7), and assume that $1 \ll \ell \ll L$. Let

$$\bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)} := \mathcal{A}_L \setminus \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)},$$

and suppose that there exists $t > 0$, such that for all $x \in \mathbb{Z}^3$,

$$\mathbb{E} \left[\left\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \right\|_{\ell^2(\mathbb{Z}^3)}^2 \right] \geq 1 - \varepsilon \tag{10}$$

is satisfied for some $\varepsilon = \varepsilon(\delta, \ell, t) > 0$. Then,

$$\mathbb{E} \left[\frac{|\bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}|}{|\mathcal{A}_L|} \right] \geq 1 - 2\varepsilon^{\frac{1}{2}} - C\ell L^{-1}$$

for a constant C which is independent of ℓ, L, ε .

Proof. We have

$$\begin{aligned} \delta_x &= \sum_{\alpha} a_x^\alpha \psi_\alpha^{(L)} \\ a_x^\alpha &= \overline{\langle \delta_x, \psi_\alpha^{(L)} \rangle} = \overline{\psi_\alpha^{(L)}(x)}, \end{aligned}$$

so that in particular,

$$\|\delta_x\|_{\ell^2(\Lambda_L)}^2 = \sum_{\alpha \in \mathcal{A}_L} |a_x^\alpha|^2 = 1. \tag{11}$$

By the Schwarz inequality, we get

$$\begin{aligned} &\left\| R_{x,\delta,\ell} e^{-itH_\omega} \delta_x \right\|_{\ell^2(\Lambda_L)}^2 \\ &\leq \left(1 + \frac{1}{\eta} \right) \left\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\ &\quad + (1 + \eta) \left\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2. \end{aligned} \tag{12}$$

For the first term on the r.h.s., we find

$$\begin{aligned}
 & \left\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & \leq \left\| R_{x,\delta,\ell} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} e^{-it\varepsilon_\alpha^{(L)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & \leq \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} |\psi_\alpha^{(L)}(x)| \|R_{x,\delta,\ell} \psi_\alpha^{(L)}\|_{\ell^2(\Lambda_L)}, \tag{13}
 \end{aligned}$$

using the *a priori* bound

$$\begin{aligned}
 & \left\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & \leq \left\| \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} e^{-it\varepsilon_\alpha^{(L)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & = \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} |a_x^\alpha|^2 \leq 1, \tag{14}
 \end{aligned}$$

which follows from $\|R_{x,\delta,\ell}\|_\infty = 1$, orthonormality of $\{\psi_\alpha^{(L)}\}_{\alpha \in \mathcal{A}_L}$ on $\ell^2(\Lambda_L)$, and (11). For the second term on the r.h.s. of (12), we likewise find

$$\begin{aligned}
 & \left\| R_{x,\delta,\ell} e^{-itH_\omega} \sum_{\alpha \in \bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & \leq \left\| \sum_{\alpha \in \bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} e^{-it\varepsilon_\alpha^{(L)}} a_x^\alpha \psi_\alpha^{(L)} \right\|_{\ell^2(\Lambda_L)}^2 \\
 & = \sum_{\alpha \in \bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} |a_x^\alpha|^2 \\
 & = \sum_{\alpha \in \bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} |\psi_\alpha^{(L)}(x)|^2. \tag{15}
 \end{aligned}$$

Averaging over $x \in \Lambda_L$, we have

$$\begin{aligned}
 & \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \|R_{x,\delta,\ell} e^{-itH_\omega} \delta_x\|_{\ell^2(\Lambda_L)}^2 \\
 & \leq (1+\eta) \frac{1}{|\Lambda_L|} \sum_{\alpha \in \tilde{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}} \sum_{x \in \Lambda_L} |\psi_\alpha^{(L)}(x)|^2 \\
 & + \left(1 + \frac{1}{\eta}\right) \frac{1}{|\Lambda_L|} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} \sum_{x \in \Lambda_L} |\psi_\alpha^{(L)}(x)| \|R_{x,\delta,\ell} \psi_\alpha^{(L)}\|_{\ell^2(\Lambda_L)} \\
 & \leq (1+\eta) \frac{1}{|\Lambda_L|} |\tilde{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}| \\
 & + \left(1 + \frac{1}{\eta}\right) \frac{1}{|\Lambda_L|} \sum_{\alpha \in \mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}} \sum_{x \in \Lambda_L} |\psi_\alpha^{(L)}(x)| \|R_{x,\delta,\ell} \psi_\alpha^{(L)}\|_{\ell^2(\Lambda_L)}. \tag{16}
 \end{aligned}$$

Let

$$S_{2\ell,L} := \left\{ x \mid \inf_{y \in \partial \Lambda_L} |x - y| \leq 2\ell \right\} \tag{17}$$

and $\tilde{\Lambda}_L := \Lambda_L \setminus S_{2\ell,L}$, such that $R_{x,\delta,\ell} \cap \partial \Lambda_L = \emptyset \ \forall x \in \tilde{\Lambda}_L$. Then,

$$\begin{aligned}
 & \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} \|R_{x,\delta,\ell} e^{-itH_\omega} \delta_x\|_{\ell^2(\Lambda_L)}^2 \\
 & = \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|R_{x,\delta,\ell} e^{-itH_\omega} \delta_x\|_{\ell^2(\mathbb{Z}^3)}^2 + O(\ell L^{-1}) \tag{18}
 \end{aligned}$$

by compactness of the support of $R_{x,\delta,\ell}$. By definition of $\mathcal{A}_{L,\varepsilon,\delta,\ell}^{(\omega)}$, the last term in (16) is bounded by $(1 + \frac{1}{\eta})\varepsilon$. Thus, recalling that $|\Lambda_L| = |\mathcal{A}_L|$,

$$\begin{aligned}
 \frac{|\tilde{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}|}{|\mathcal{A}_L|} & \geq \frac{1}{1+\eta} \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|R_{x,\delta,\ell} e^{-itH_\omega} \delta_x\|_{\ell^2(\mathbb{Z}^3)}^2 \\
 & - \frac{1 + \frac{1}{\eta}}{1 + \eta} \varepsilon - c\ell L^{-1}. \tag{19}
 \end{aligned}$$

Taking expectations, using (10), and choosing $\eta = \varepsilon^{\frac{1}{2}}$, the claim follows. ■

Lemma 3.2. Under the same assumptions as in Lemma 3.1,

$$\mathbb{P} \left[\liminf_{L \rightarrow \infty} \frac{|\bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}|}{|\mathcal{A}_L|} \geq 1 - 2\varepsilon^{\frac{1}{2}} \right] = 1. \tag{20}$$

Proof. We consider the family of translation operators $\tau_x : \omega_y \mapsto \omega_{x+y}$, for $x \in \mathbb{Z}^3$, which acts ergodically on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.⁽¹⁾

Let U_{τ_x} denote the unitary translation operator $(U_{\tau_x}\phi)(y) = \phi(x+y)$ on $\ell^2(\mathbb{Z}^3)$. Then, clearly,

$$U_{\tau_x}^* H_\omega U_{\tau_x} = -\frac{1}{2} \Delta + \lambda V_{\tau_{-x}\omega} = H_{\tau_{-x}\omega} \tag{21}$$

with $V_{\tau_x\omega}(y) = V_\omega(x+y)$, and

$$\begin{aligned} & \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|R_{x,\delta,\ell} e^{-itH_\omega} \delta_x\|_{\ell^2(\mathbb{Z}^3)}^2 \\ &= \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|(U_{\tau_x}^* R_{x,\delta,\ell} U_{\tau_x})(U_{\tau_x}^* e^{-itH_\omega} U_{\tau_x}) \delta_0\|_{\ell^2(\mathbb{Z}^3)}^2 \\ &= \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|R_{0,\delta,\ell} e^{-itH_{\tau_{-x}\omega}} \delta_0\|_{\ell^2(\mathbb{Z}^3)}^2, \end{aligned} \tag{22}$$

by unitarity of U_{τ_x} . By the Birkhoff–Khinchin ergodic theorem, applied to the random variable $X(\omega) := \|R_{0,\delta,\ell} e^{-itH_\omega} \delta_0\|_{\ell^2(\mathbb{Z}^3)}^2$, we obtain, for fixed λ ,

$$\liminf_{L \rightarrow \infty} \frac{1}{|\tilde{\Lambda}_L|} \sum_{x \in \tilde{\Lambda}_L} \|R_{0,\delta,\ell} e^{-itH_{\tau_{-x}\omega}} \delta_0\|_{\ell^2(\mathbb{Z}^3)}^2 = \mathbb{E} \left[\|R_{0,\delta,\ell} e^{-itH_\omega} \delta_0\|_{\ell^2(\mathbb{Z}^3)}^2 \right] \tag{23}$$

with probability one. We note here that clearly, the left hand side of (10) is independent of $x \in \mathbb{Z}^3$. Therefore, (10), (19) and (23) imply

$$\mathbb{P} \left[\liminf_{L \rightarrow \infty} \frac{|\bar{\mathcal{A}}_{L,\varepsilon,\delta,\ell}^{(\omega)}|}{|\mathcal{A}_L|} \geq 1 - \frac{\eta}{1+\eta} - \frac{2 + \frac{1}{\eta}}{1+\eta} \varepsilon \right] = 1, \tag{24}$$

and choosing $\eta = \varepsilon^{\frac{1}{2}}$, the claim follows. ■

From here on, we will write $\| \cdot \|_2 \equiv \| \cdot \|_{\ell^2(\mathbb{Z}^3)}$.

To conclude the proof of Theorem 2.1, we use the key Lemma 3.3 below, which provides the lower bound

$$\mathbb{E} \left[\| R_{x,\delta,\lambda^{-2}} e^{-it(\delta,\lambda)H_\omega} \delta_x \|_2^2 \right] \geq 1 - C\delta^{\frac{3}{7}} \tag{25}$$

for the choice $t(\delta, \lambda) = \delta^{\frac{6}{7}} \lambda^{-2}$, $\lambda^{\frac{14}{15}} < \delta < 1$, and a constant c that is independent of x, λ and δ . Thus, choosing $\varepsilon = \delta^{\frac{3}{7}}$, (25) immediately implies Theorem 2.1.

Lemma 3.3. Let

$$t(\delta, \lambda) = \delta^{\frac{6}{7}} \lambda^{-2}, \tag{26}$$

and $H_0 := -\frac{1}{2}\Delta$. Then, for λ sufficiently small, $0 < \delta < 1$, and all $x \in \mathbb{Z}^3$, the free evolution term satisfies

$$\| R_{x,\delta,\lambda^{-2}} e^{-it(\delta,\lambda)H_0} \delta_x \|_2 \geq 1 - C\delta^{\frac{3}{7}}, \tag{27}$$

while

$$\mathbb{E} \left[\| R_{x,\delta,\lambda^{-2}} [e^{-it(\delta,\lambda)H} - e^{-it(\delta,\lambda)H_0}] \delta_x \|_2^2 \right] \leq C'\delta^{\frac{6}{7}} + \delta^{-\frac{6}{49}} \lambda^{\frac{2}{7}} \tag{28}$$

for finite positive constants C, C' that are independent of x, λ and δ .

Proof. We may assume that $x = 0$. Let

$$\ell_2 := \ell, \quad \ell_1 := \delta\ell. \tag{29}$$

We recall that $(R_{0,\delta,\ell})^2 = K_{\ell_2}^2 - 2K_{\ell_1} + K_{\ell_1}^2$. To bound $\|K_{\ell_2} e^{-itH_0} \delta_0\|_2$, we note that

$$\begin{aligned} & \left| e^{-ite_\Delta(p)} - \int_{\mathbb{T}^3} dk \hat{K}_{\ell_2}(p-k) e^{-ite_\Delta(k)} \right| \\ & \leq C \sup_{|p-k| \leq \gamma} \left| e^{-ite_\Delta(p)} - e^{-ite_\Delta(k)} \right| \\ & \quad + 2 \int_{\mathbb{T}^3} dk |\hat{K}_{\ell_2}(k)| \chi(|p-k| \geq \gamma) \\ & \leq C\gamma t + C\gamma^{-1} \ell_2^{-1}, \end{aligned} \tag{30}$$

owing to

$$|\hat{K}_{\ell_2}(k)| \leq C \prod_{j=1}^3 \frac{\ell_2}{1 + \|k_j\|_{\mathbb{Z}}^2 \ell_2^2}, \quad \int_{\mathbb{T}^3} dk \hat{K}_{\ell_2}(k) = 1, \tag{31}$$

where $\|r\|_{\mathbb{Z}} := \text{dist}(r, \mathbb{Z})$ for $r \in \mathbb{R}$, which are basic properties of the Fejér kernel. Thus, with

$$t = \delta^\alpha \lambda^{-2}, \quad \ell_2 = \lambda^{-2}, \quad \gamma = t^{-\frac{1}{2}} \ell_2^{-\frac{1}{2}}, \tag{32}$$

we find

$$(30) \leq C \delta^{\frac{\alpha}{2}}. \tag{33}$$

Hence,

$$\begin{aligned} \left\| K_{\ell_2} e^{-itH_0} \delta_0 \right\|_2^2 &= \int_{\mathbb{T}^3} dp \left| e^{-ite_{\Delta}(p)} + O(\delta^{\frac{\alpha}{2}}) \right|^2 \\ &\geq 1 - C \delta^{\frac{\alpha}{2}}. \end{aligned} \tag{34}$$

Next, we consider

$$\left\| |K_{\ell_1}|^{\frac{a}{2}} e^{-itH_0} \delta_0 \right\|_2^2 = \sum_{y \in \mathbb{Z}^3} |K_{\ell_1}(y)|^a \left| (e^{-itH_0} \delta_0)(y) \right|^2, \tag{35}$$

where $a = 1, 2$, and

$$(e^{-itH_0} \delta_0)(y) = \int_{\mathbb{T}^3} dk e^{-i(te_{\Delta}(k) - 2\pi ky)}. \tag{36}$$

We observe that the kinetic energy $e_{\Delta} : \mathbb{T}^3 \rightarrow [0, 6]$ is a real analytic Morse function with eight critical points in the corners of the subcube $[0, \frac{1}{2}]^3 \subset \mathbb{T}^3 = [0, 1]^3$. Each of the remaining critical points in $[0, 1]^3 \setminus [0, \frac{1}{2}]^3$ is identified with one of the latter by symmetry. The Hessians are diagonal and have entries of modulus $4\pi^2$.

We bound |(36)| by a stationary phase estimate. For $|y| \leq C\ell_1$, $\ell_1 = \delta\lambda^{-2}$ and $t = \delta^\alpha \lambda^{-2}$, it is clear that

$$\nabla(e_{\Delta} - 2\pi t^{-1} \langle y, \cdot \rangle)(k^*) = 0 \tag{37}$$

implies

$$|\nabla e_\Delta(k^*)| < C\delta^{1-\alpha}.$$

It follows that for each of the eight critical points of e_Δ , there is precisely one k^* satisfying (37) in its $\delta^{1-\alpha}$ -vicinity, given that $\delta^{1-\alpha}$ is sufficiently small. Correspondingly, $\text{Hess}[e_\Delta](k^*)$ is in each of these cases non-degenerate, with eigenvalues of modulus $O(1)$.

We introduce a smooth partition of unity $\sum \phi_j = 1$ on $[0, 1]^3$, $j \in \{1, \dots, 8\}$, continued over the boundary of by periodicity, in a manner that each $\text{supp}\phi_j$ is centered at one critical point of e_Δ . By the above, a stationary phase estimate yields

$$\sup_{y \in \text{supp}K_{\ell_1}} \left| \sum_j \int_{\mathbb{T}^3} dk \phi_j(k) e^{-i(te_\Delta(k) - 2\pi ky)} \right| \leq Ct^{-3/2}.$$

Consequently,

$$|(35)| \leq C\ell_1^3 t^{-3} = C\delta^{3(1-\alpha)}, \tag{38}$$

and optimizing the bounds, we find $\alpha = \frac{6}{7}$.

Our strategy to prove (28) employs a modification of the methods of Erdős and Yau from ref. 3. Thereby, we invoke a Duhamel expansion with remainder term, and control the expectation by classifying all contraction types occurring in the products of the random potential. The remainder term is bounded by exploiting the rarity of the event that a large number of collisions occurs in a small time interval.

As a result, we obtain

$$\mathbb{E} \left[\left\| R_{x,\delta,\lambda^{-2}} [e^{-itH} - e^{-itH_0}] \delta_x \right\|_2^2 \right] \leq C_1 \lambda^2 t + t^{-\frac{1}{7}} \tag{39}$$

for a constant C_1 that is independent of x , λ and δ . This implies (28) for the asserted choice of t . The proof of (39) will occupy Sections 3.1–3.7. ■

3.1. Expectation of Products of Random Potentials

We shall to begin with consider the expectation of products of random potentials. The pair correlation is given by the Kronecker delta

$$\mathbb{E}[\omega_{x_1} \omega_{x_2}] = \delta_{x_1, x_2},$$

and we recall that by our assumptions on ω_x , the m -point correlation is zero for any odd m . The fourth order correlation yields

$$\begin{aligned} \mathbb{E}[\omega(x_1)\omega(x_2)\omega(x_3)\omega(x_4)] &= (1 - \delta_{x_1,x_3})\delta_{x_1,x_2}\delta_{x_3,x_4} + (1 - \delta_{x_1,x_2})\delta_{x_1,x_3}\delta_{x_2,x_4} \\ &\quad + (1 - \delta_{x_1,x_3})\delta_{x_1,x_4}\delta_{x_2,x_3} + \tilde{c}_4\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3} \end{aligned} \tag{40}$$

$$\begin{aligned} &= \delta_{x_1,x_2}\delta_{x_3,x_4} + \delta_{x_1,x_3}\delta_{x_2,x_4} + \delta_{x_1,x_4}\delta_{x_2,x_3} \\ &\quad + (\tilde{c}_4 - 3)\delta_{x_1,x_2}\delta_{x_3,x_4}\delta_{x_1,x_3}. \end{aligned} \tag{41}$$

The operation applied in passing from (40) to (41) will be referred to as *Wick ordering*. By a renormalization of the fourth order moment of ω_x ,

$$\tilde{c}_4 \rightarrow c_4 := \tilde{c}_4 - 3\tilde{c}_2^2$$

(where $\tilde{c}_2 = 1$), it decomposes (40) into independent terms.

For the Fourier transformed random potentials $\hat{\omega}(k) := \sum_x \omega_x e^{2\pi i k x}$, one obtains exact Dirac delta distributions for the Wick ordered expression (41),

$$\begin{aligned} \mathbb{E}[\hat{\omega}(k_1)\hat{\omega}(k_2)\hat{\omega}(k_3)\hat{\omega}(k_4)] &= \delta(k_1 + k_2)\delta(k_3 + k_4) + \delta(k_1 + k_3)\delta(k_2 + k_4) \\ &\quad + \delta(k_1 + k_4)\delta(k_2 + k_3) + c_4\delta(k_1 + k_2 + k_3 + k_4). \end{aligned}$$

We note that this is not the case for the individual summands in (40) prior to Wick ordering. The same statement applies to all higher order correlations.

The Wick ordered product of an arbitrary even number of random potentials is determined as follows. We introduce, for $n, n' \in \mathbb{N}$ with $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$, the set

$$\mathcal{V}_{n,n'} := \left\{ 1, \dots, n, n+2, \dots, n+n'+1 \right\}.$$

In our later discussion, $\mathcal{V}_{n,n'}$ labels a linearly ordered set of $n+n'$ random potentials that are, in frequency space, subdivided into a group of n' copies of \hat{V}_ω , and a group of n copies of \hat{V}_ω (the complex conjugate). The label $n+1$ excluded here is reserved for a distinguished point that is not attributed to a random potential. We note again that the case $n+n' \in 2\mathbb{N}_0 + 1$ is trivial since all odd moments of V_ω vanish.

Definition 3.1. For $\bar{n} = \frac{n+n'}{2} \in \mathbb{N}$, let

$$\Pi_{n,n'} := \bigcup_{m=1}^{\bar{n}} \left\{ \{S_j\}_{j=1}^m \mid |S_j| \in 2\mathbb{N}; \mathcal{V}_{n,n'} = \bigcup_{j=1}^m S_j; S_j \cap S_{j'} = \emptyset \text{ if } j \neq j' \right\} / \mathfrak{S}_m$$

denote the set of partitions of $\mathcal{V}_{n,n'}$ into disjoint subsets S_j (referred to as blocks) of size $|S_j| \in 2\mathbb{N}$, where \mathfrak{S}_m is the m -th symmetric group. Two partitions $\pi = \{S_j\}_{j=1}^m, \pi' = \{S'_j\}_{j=1}^m$ are equivalent, $\pi = \pi'$, if $\exists \sigma \in \mathfrak{S}_m$ such that $S_j = S'_{\sigma(j)}$ for all $j \in \{1, \dots, m\}$. A partition $\pi \in \Pi_{n,n'}$ will also be referred to as a contraction (corresponding to contractions among random potentials).

The number of $\pi \in \Pi_{n,n'}$ consisting of m blocks is given by

$$\begin{aligned} B_{\bar{n}}(m) &:= \sharp \left\{ \{S_j\}_{j=1}^m \mid \bigcup_{j=1}^m S_j = \mathcal{V}_{n,n'}; |S_j| \in 2\mathbb{N}; S_i \cap S_j = \emptyset \text{ if } i \neq j \right\} / \mathfrak{S}_m \\ &= \sum_{r=1}^{\bar{n}} \sum_{1 \leq j_1, \dots, j_r \leq \bar{n}} \sum_{1 \leq l_1 < \dots < l_r \leq \bar{n}} \frac{\delta_{m, |j|} \delta_{\bar{n}, \langle j, l \rangle}}{(2\bar{n})!} \frac{1}{(j_1!) \dots (j_r!)}, \end{aligned} \tag{42}$$

where $\underline{j} := (j_1, \dots, j_r), |j| := \sum_{i=1}^r j_i$, and $\langle \underline{j}, \underline{l} \rangle := \sum_{i=1}^r j_i l_i$ for every r . Here, j_i is the number of blocks of size $2l_i$. The factor $\frac{1}{j_i!}$ arises because the order is irrelevant, according to which blocks of the same size are counted. We note that the number of partitions into products of pair correlators (that is, $r = 1, j = \bar{n}, l = 1$) is

$$B_{\bar{n}}(\bar{n}) = 1 \cdot 3 \cdot \dots \cdot (2\bar{n} - 1) < 2^{\bar{n}} (\bar{n})!.$$

On the other hand, it is clear that

$$B_{\bar{n}}(m) < \sum_{0 \leq s_1, \dots, s_m \leq 2\bar{n}} \delta_{2\bar{n}, \sum_{i=1}^m s_i} \frac{(2\bar{n})!}{(s_1!) \dots (s_m!)} = m^{2\bar{n}},$$

hence for non-pairing contractions, i.e. $m < \bar{n}$,

$$\sum_{m=1}^{\bar{n}-1} B_{\bar{n}}(m) < \bar{n}^{2\bar{n}+1}.$$

This trivial estimate will suffice for our purposes.

For $S \subset \mathcal{V}_{n,n'}$, with $|S| \in 2\mathbb{N}$, we define

$$\delta(x_S) := \sum_{y \in \mathbb{Z}^3} \prod_{j \in S} \delta_{x_j, y},$$

where $x_S := (x_j)_{j \in S}$. Then,

$$\begin{aligned} \mathbb{E} \left[\prod_{j \in \mathcal{V}_{n,n'}} V_\omega(x_j) \right] &= \sum_{m=1}^{\bar{n}} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi = \{S_j\}_{j=1}^m}} \left(\prod_{j=1}^m \tilde{c}_{|S_j|} \delta(x_{S_j}) \right) \\ &\quad \times \prod_{1 \leq i < j \leq m} (1 - \delta_{x_{\mu(i)}, x_{\mu(j)}}), \end{aligned} \tag{43}$$

where for definiteness, $\mu(i) := \min \{q | q \in S_i\}$ (clearly, one could choose any arbitrary element of S_i). Due to the second product, the factors in $\prod c_{|S_j|} \delta_{S_j}$ are not independent. We note that

$$\delta(x_{S_i}) \delta(x_{S_j}) \delta_{x_{\mu(i)}, x_{\mu(j)}} = \delta(x_{S_i \cup S_j}), \tag{44}$$

where of course, $|S_i \cup S_j| = |S_i| + |S_j|$. Therefore, expanding $\prod (1 - \delta_{x_{\mu(i)}, x_{\mu(j)}})$ in (43), using (44) recursively, and collecting all terms belonging to the same blocks, we find

$$\mathbb{E} \left[\prod_{j \in \mathcal{V}_{n,n'}} V_\omega(x_j) \right] = \sum_{m=1}^{\bar{n}} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi = \{S_j\}_{j=1}^m}} \prod_{j=1}^m c_{|S_j|} \delta(x_{S_j}), \tag{45}$$

where the cumulant formula

$$\begin{aligned} c_{2k} &= \sum_{m=1}^k \sum_{r=1}^k \sum_{1 \leq j_1, \dots, j_r \leq k} \sum_{1 \leq l_1 < \dots < l_r \leq k} \delta_{m, |j|} \delta_{k, (j, l)} \\ &\quad \times \frac{(-1)^{m-1} (2k)!}{((2l_1)!)^{j_1} \dots ((2l_r)!)^{j_r}} \frac{\tilde{c}_{2l_1}^{j_1} \dots \tilde{c}_{2l_r}^{j_r}}{(j_1!) \dots (j_r!)} \end{aligned}$$

determines the renormalized moments of ω_x . Thus, (45) decomposes the expectation value into the sum of all possible products of correlators, which are now mutually independent. We observe that by (3),

$$\begin{aligned}
 |c_{2k}| &\leq (2k)! k \sum_{r=1}^k \left(\sum_{1 \leq j_1, \dots, j_r \leq k} \frac{c_V^{j_1 + \dots + j_r}}{(j_1! \cdots (j_r!))} \right) \left(\sum_{1 \leq l_1 < \dots < l_r \leq k} 1 \right) \\
 &\leq (2k)! 2k \sum_{r=1}^k \frac{(k e^{c_V})^r}{r!} \\
 &\leq k^{k+1} \frac{(2k)!}{k!} e^{k e^{c_V}} \\
 &\leq (2k e^{\frac{1}{2} e^{c_V}})^{2k+1}.
 \end{aligned} \tag{46}$$

Then, the expectation of the full product of random potentials decomposes into

$$\mathbb{E} \left[\prod_{j \in \mathcal{V}_{n,n'}} \hat{V}_\omega(k_j) \right] = \sum_{m=1}^{\tilde{n}} \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi = \{S_j\}_{j=1}^m}} \left(\prod_{j=1}^m c_{|S_j|} \right) \delta \left(\sum_{i \in S_j} k_i \right) \tag{47}$$

in momentum space, where $c_{|S_j|}$ are the renormalized moments of ω_x .

3.2. Duhamel Expansion

Our aim is to prove the bound (28) by classifying and estimating the integrals corresponding to all contractions occurring on the left hand side of (28).

To this end, we invoke the Duhamel expansion of $\phi_t = e^{-itH_\omega} \delta_x$. For $N \in \mathbb{N}$ large, which remains to be determined, it is given by

$$\begin{aligned}
 (e^{-itH_\omega} \delta_x - e^{-itH_0} \delta_x)(y) &= \left((-i\lambda) \int_0^t ds e^{-i(t-s)H_\omega} V_\omega e^{-isH_0} \delta_x \right)(y) \\
 &= \sum_{n=1}^N \phi_{n,t}(y) + R_{N,t}(y).
 \end{aligned} \tag{48}$$

Writing

$$\left[\prod_{j=0}^n ds_j \right]_t := ds_0 \cdots ds_n \delta \left(\sum_{j=0}^n s_j - t \right)$$

for brevity, the Fourier transform of the n th Duhamel term is given by

$$\begin{aligned}
 \hat{\phi}_{n,t}(k_0) &= (-i\lambda)^n \int_{\mathbb{R}_+^{n+1}} \left[\prod_{j=0}^n ds_j \right]_t \int_{(\mathbb{T}^3)^n} e^{2\pi i k_n \cdot x} e^{-i \sum_{j=0}^n s_j e_{\Delta}(k_j)} \\
 &\quad \times \prod_{j=1}^n \hat{V}_{\omega}(k_j - k_{j-1}) dk_j \\
 &= \frac{i e^{\varepsilon t} (-i\lambda)^n}{2\pi} \int_{\mathbb{R}} d\alpha e^{-i\alpha t} \int_{(\mathbb{T}^3)^n} e^{2\pi i k_n \cdot x} \prod_{j=0}^n \frac{1}{e_{\Delta}(k_j) - \alpha - i\varepsilon} \\
 &\quad \times \prod_{l=1}^n \hat{V}_{\omega}(k_l - k_{l-1}) dk_l, \tag{49}
 \end{aligned}$$

where we shall choose

$$\varepsilon = t^{-1} \tag{50}$$

in all that follows. α is an energy parameter, and the multiplication operators $\frac{1}{e_{\Delta}(k_j) - \alpha - i\varepsilon}$ are the Fourier transformed resolvents of $-\frac{1}{2}\Delta$ (which will also be referred to as particle propagators). The explicit formula for the remainder term $R_{N,t}$ can be found in (144) below.

We note that in this analysis, $\varepsilon = t^{-1}$ and λ will be the small parameters of the theory, which will ultimately be related through $\varepsilon = C\lambda^2$.

Let $\mathbb{H}_- := \{z \in \mathbb{C} \mid \text{Im}(z) \leq 0\}$. The integrand is analytic in α , and it is not hard to see that the path of the α -integration can, for any fixed $n \in \mathbb{N}$, be deformed away from \mathbb{R} into the closed contour

$$I = I_{\mathbb{R}} \cup I_{\mathbb{H}_-} \tag{51}$$

with

$$\begin{aligned}
 I_{\mathbb{R}} &:= [-1, 7] \\
 I_{\mathbb{H}_-} &:= ([-1, 7] - i) \cup (-1 - i(0, 1]) \cup (7 - i(0, 1]) \subset \mathbb{H}_-,
 \end{aligned}$$

which encloses $\text{spec}\{-\frac{1}{2}\Delta - i\varepsilon\} = [0, 6] - i\varepsilon$. Consequently,

$$\begin{aligned}
 \hat{\phi}_{n,t}(k_0) &= \frac{i e^{\varepsilon t} (-i\lambda)^n}{2\pi} \int_I d\alpha e^{-i\alpha t} \int_{(\mathbb{T}^3)^n} dk_1 \dots dk_n e^{2\pi i k_n \cdot x} \\
 &\quad \times \left[\prod_{j=0}^n \frac{1}{e_{\Delta}(k_j) - \alpha - i\varepsilon} \right] \prod_{l=1}^n \hat{V}_{\omega}(k_l - k_{l-1}),
 \end{aligned}$$

where the loop I is taken in the clockwise direction.

Using the Schwartz inequality,

$$\text{l.h.s. of (39)} \leq 2\mathbb{E}\left[\left\|\sum_{n=1}^N \phi_{n,t}\right\|_2^2\right] + 2\mathbb{E}\left[\left\|R_{N,t}^\varepsilon\right\|_2^2\right]. \tag{52}$$

For $1 \leq n, n' \leq N$, and $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}[\langle \phi_{n',t}, \phi_{n,t} \rangle] &= \frac{e^{2\varepsilon t} \lambda^{2n}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ &\times \int_{(\mathbb{T}^3)^{2\bar{n}+2}} \left[\prod_{j=0}^n \prod_{l=0}^{n'} dk_j d\tilde{k}_l \right] \delta(k_0 - \tilde{k}_0) e^{-2\pi i(k_n - \tilde{k}_{n'}) \cdot x} \\ &\times \prod_{j=0}^n \prod_{l=0}^{n'} \frac{1}{e_\Delta(k_j) - \alpha - i\varepsilon} \frac{1}{e_\Delta(\tilde{k}_l) - \beta + i\varepsilon} \\ &\times \mathbb{E} \left[\prod_{j=1}^n \prod_{l=1}^{n'} \hat{V}_\omega(k_j - k_{j-1}) \overline{\hat{V}_\omega(\tilde{k}_l - \tilde{k}_{l-1})} \right], \end{aligned} \tag{53}$$

where \bar{I} is the complex conjugate of I , and taken in the counterclockwise direction by the variable β .

Introducing new variables

$$\underline{p} = (p_0, \dots, p_n, p_{n+1}, \dots, p_{2\bar{n}+1}) \tag{54}$$

$$:= (\tilde{k}_{n'}, \dots, \tilde{k}_0, k_0, \dots, k_n) \tag{55}$$

and

$$(\alpha_j, \sigma_j) = \begin{cases} (\alpha, 1) & 0 \leq j \leq n \\ (\beta, -1) & n < j \leq 2n + 1, \end{cases} \tag{56}$$

we can write

$$\begin{aligned} \mathbb{E}[\langle \phi_{n',t}, \phi_{n,t} \rangle] &= \frac{e^{2\varepsilon t} \lambda^{2\bar{n}}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ &\times \int_{(\mathbb{T}^3)^{2\bar{n}+2}} d\underline{p} \delta(p_n - p_{n+1}) \prod_{j=0}^{2\bar{n}+1} \frac{1}{e_{\Delta}(p_j) - \alpha_j - i\sigma_j \varepsilon} \\ &\times \mathbb{E} \left[\prod_{\substack{i=1 \\ i \neq n+1}}^{2\bar{n}+1} \hat{V}_{\omega}(p_i - p_{i-1}) \right] e^{-2\pi i(p_0 - p_{2\bar{n}+1}) \cdot x}, \end{aligned} \tag{57}$$

noting that $\overline{\hat{V}(k)} = \hat{V}(-k)$.

Let $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n'}$ denote a partition. Let

$$\delta_{S_j}(\underline{p}) := \delta \left(\sum_{i \in S_j} (p_i - p_{i-1}) \right)$$

and

$$\delta_{\pi}(\underline{p}) := \prod_{j=1}^m \delta_{S_j}(\underline{p}).$$

Then, the contribution to (57) corresponding to π is given by the singular integral

$$\begin{aligned} \text{Amp}[\pi] &:= \frac{e^{2\varepsilon t} \lambda^{2\bar{n}}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \int_{(\mathbb{T}^3)^{2\bar{n}+2}} d\underline{p} \delta(p_n - p_{n+1}) \\ &\times e^{-2\pi i(p_0 - p_{2\bar{n}+1}) \cdot x} \left(\prod_{l=1}^m c_{|S_l|} \right) \delta_{\pi}(\underline{p}) \\ &\times \prod_{j=0}^{2\bar{n}+1} \frac{1}{e_{\Delta}(p_j) - \alpha_j - i\sigma_j \varepsilon}, \end{aligned} \tag{58}$$

referred to as the (*Feynman*) *amplitude* corresponding to π . The expectation (57) is obtained from summing the amplitudes $\text{Amp}[\pi]$ over all partitions $\pi \in \Pi_{n,n'}$.

3.3. The Graph Representation of Contractions

To estimate the expectation (57), it is necessary to classify the singular integrals $\text{Amp}[\pi]$, whose size depends on the structure of π . For this combinatorial problem, it is natural to represent π , encoded in the delta distributions δ_π in (58), by (*Feynman*) *graphs*. We shall use the following prescription, cf. Fig. 1. We draw two parallel solid ‘particle lines’, joined together at one end, accounting for $\delta(p_n - p_{n+1})$, containing n , respectively n' vertices, where $n + n' \in 2\mathbb{N}$. Every pairing contraction is depicted by a dashed line joining the respective vertices. The higher correlation contractions corresponding to δ_{S_j} are represented by $|S_j| \in 2\mathbb{N}$ dashed lines connecting the corresponding vertices to one mutual vertex that is disjoint from the particle lines. Any (solid) edge that lies on a particle line refers to a particle propagator.

Let G_π denote the graph associated to a partition $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n'}$. The set of vertices of the graph G_π is denoted by $V(G_\pi)$, and the set of edges as $E(G_\pi)$. $V(G_\pi)$ is the union $V(G_\pi) = V_p(G_\pi) \cup V_{hc}(G_\pi)$, where $V_p(G_\pi) = \mathcal{V}_{n,n'}$ is the $n + n'$ -subset of vertices on the particle line, and $V_{hc}(G_\pi)$ is the subset of vertices disjoint from the particle lines, which are associated to correlations of higher order than two. In the product of Kronecker deltas (45) in the position space picture, the elements of $V_p(G_\pi)$ correspond to the sites x_i of random potentials, while the elements of $V_{hc}(G_\pi)$ correspond to the dummy summation variables y_j .

Definition 3.2. A contraction $\pi = \{S_j\} \in \Pi_{n,n'}$ is called a pairing contraction if $m = \bar{n}$, so that $|S_j| = 2$ for all j . Otherwise, π is called a higher (order) correlation contraction, or a type III contraction (cf. Definition 3.3 below).

It is in fact necessary to introduce the following finer classification of families of contractions, see ref. 3.

Definition 3.3. The delta distributions associated to partitions $\pi \in \Pi_{n,n'}$ of the set $\mathcal{V}_{n,n'}$ are classified into the following types, according to the corresponding subgraph structure. A delta distribution $\delta_S(\underline{p}) = \delta(p_i - p_{i-1} + p_j - p_{j-1})$ associated to a pairing S with $|S| = 2$ is of

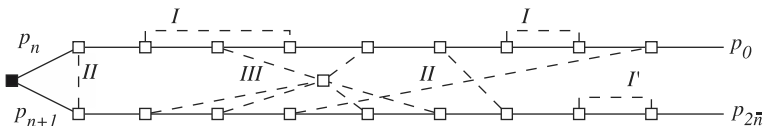


Fig. 1. A graph containing type I, I', II, and III contractions.

TYPE I if $i, j \leq n$.

TYPE I' if $i, j > n$.

TYPE II if $i \leq n$, but $j \geq n + 2$.

A delta function δ_S is of

TYPE III if $|S| \geq 4$, that is, if it is not associated to a pairing contraction.

Hence, a partition of $\mathcal{V}_{n,n'}$ is of type III if it contains a type III delta distribution.

Definition 3.4. A pairing contraction $\pi \in \Pi_{n,n'}$ is called crossing if δ_π contains two delta distributions $\delta(p_{i_1} - p_{i_1-1} + p_{j_1} - p_{j_1-1})$ and $\delta(p_{i_2} - p_{i_2-1} + p_{j_2} - p_{j_2-1})$, with $j_r > i_r$, such that $i_1 - i_2$ and $j_1 - j_2$ have the same signs.

Definition 3.5. A non-crossing pairing contraction $\pi \in \Pi_{n,n'}$ is called nested if δ_π contains two delta distributions $\delta(p_{i_1} - p_{i_1-1} + p_{j_1} - p_{j_1-1})$ and $\delta(p_{i_2} - p_{i_2-1} + p_{j_2} - p_{j_2-1})$, with $j_r > i_r$, both either of type I or of type I', such that $i_1 - i_2$ and $j_1 - j_2$ have opposite signs.

Definition 3.6. A non-crossing and non-nested pairing contraction is called simple. A simple pairing contraction is called a ladder graph if all of its associated delta functions are of type II.

Assume $\pi \in \Pi_{n,n'}$ is a pairing contraction. A spanning tree T of G_π is a connected tree graph that contains $V(G_\pi)$. We denote the set of edges contained in T by E_T , and refer to the corresponding momenta as tree momenta. The momenta corresponding to the edges in the complement $E_L = E'_T$ are referred to as loop momenta. Adding any edge of E_L to the spanning tree T produces a loop.

Definition 3.7. A spanning tree T of G_π with $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n'}$ a pairing contraction is called complete if it contains all contraction lines, and the edge corresponding to the momentum p_n , but not the one corresponding to the momentum p_{n+1} .

3.4. Simple Pairing Contractions

It is our aim to estimate $|\text{Amp}[\pi]|$ for each type of contractions $\pi \in \Pi_{n,n'}$ listed above. We shall proceed by first discussing simple pairings, then crossing and nested pairings, and finally type III contractions.

Similarly as in ref. 3, we will find that the amplitudes $\{\text{Amp}[\pi] | \pi \text{ simple}\}$ completely dominate over those associated to all other contraction classes (notably even in the presence of type III contractions).

3.4.1. The Ladder Graph

The simplest member in the class of simple pairings in $\Pi_{n,n}$ is the *ladder graph*. It corresponds to the pairing $\pi = \{S_j\}_{j=1}^n \in \Pi_{n,n}$, with $S_j = \{j, 2n+2-j\}$, such that $|S_j|=2$, and

$$\begin{aligned} \text{Amp}[\pi] &= \frac{e^{2\varepsilon t} \lambda^{2n}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \int_{(\mathbb{T}^3)^{2n+2}} d\underline{p} \delta(p_n - p_{n+1}) \\ &\quad \times e^{-2\pi i(p_0 - p_{2n+1}) \cdot x} \prod_{l=0}^{2n+1} \frac{1}{e_\Delta(p_l) - \alpha_l - i\sigma_l \varepsilon} \\ &\quad \times \prod_{j=1}^n \delta\left((p_j - p_{j-1}) + (p_{2n+2-j} - p_{2n+1-j})\right), \end{aligned} \tag{59}$$

cf. (58).

The following L^∞ and L^1 resolvent estimates will be used extensively in the sequel.

Lemma 3.4. Let $0 < \varepsilon \ll 1$. Then,

$$\sup_{\alpha \in I} \sup_{p \in \mathbb{T}^3} \frac{1}{|e_\Delta(p) - \alpha - i\varepsilon|} \leq \frac{1}{\varepsilon} \tag{60}$$

$$\sup_{\alpha \in I} \int_{\mathbb{T}^3} dp \frac{1}{|e_\Delta(p) - \alpha - i\varepsilon|} \leq C \log \frac{1}{\varepsilon} \tag{61}$$

$$\sup_{p \in \mathbb{T}^3} \int_I |d\alpha| \frac{1}{|e_\Delta(p) - \alpha - i\varepsilon|} \leq C \log \frac{1}{\varepsilon}$$

for finite constants C that are uniform in ε .

Proof. Since by definition of I , $\inf_{p \in \mathbb{T}^3} \text{dist}(e_\Delta(p) - i\varepsilon, I) = \varepsilon$, and since $|I|$ is finite, (60) and the second estimate in (61) are evident.

To prove the first estimate in (61), we first show that the measure of the isoenergy surface

$$\Sigma_\alpha := \{p \in \mathbb{T}^3 \mid e_\Delta(p) = \alpha\}$$

is uniformly bounded with respect to $\alpha \in I \cap \mathbb{R}$. We note that for $\text{Im}(\alpha) \neq 0$, the asserted bound is trivial. For $p = (p_1, p_2, p_3) \in \mathbb{T}^3$, let

$$e_{2D}(\underline{p}) := \sum_{j=1,2} (1 - \cos 2\pi p_j), \quad \underline{p} := (p_1, p_2) \tag{62}$$

denote the Fourier transform of the 2-D nearest neighbor Laplacian, and

$$s_r := \{ \underline{p} \in \mathbb{T}^2 \mid e_{2D}(\underline{p}) = r \} \tag{63}$$

the corresponding level curves. Then,

$$\text{mes}\{s_r\} = \int_{[0,1]^2} d\underline{p} \delta(e_{2D}(\underline{p}) - r) \tag{64}$$

is easily seen to be uniformly bounded,

$$\sup_r \text{mes}\{s_r\} < C . \tag{65}$$

Therefore,

$$\begin{aligned} \text{mes}\{\Sigma_\alpha\} &= \int_0^1 dp_3 \int_{\mathbb{T}^2} d\underline{p} \delta(e_{2D}(\underline{p}) + (1 - \cos 2\pi p_3) - \alpha) \\ &= 2 \int_0^1 dk (1 - k^2)^{-\frac{1}{2}} \int_{\mathbb{T}^2} d\underline{p} \delta(e_{2D}(\underline{p}) + 1 - k - \alpha) \\ &\leq 2 \int_0^1 dk (1 - k^2)^{-\frac{1}{2}} \sup_r \text{mes}\{s_r\} \leq C , \end{aligned} \tag{66}$$

uniformly in α . Thus, defining

$$R_j(\alpha, \varepsilon) := \left\{ p \in \mathbb{T}^3 \mid 2^j \varepsilon < e_\Delta(p) \leq 2^{j+1} \varepsilon \right\}, \tag{67}$$

we have

$$\begin{aligned} \text{mes}\{R_j(\alpha, \varepsilon)\} &= \int_{2^j \varepsilon}^{2^{j+1} \varepsilon} d\alpha' \int_{\mathbb{T}^3} dp \delta(\alpha' - e_\Delta(p)) \\ &= \int_{2^j \varepsilon}^{2^{j+1} \varepsilon} d\alpha' \text{mes}\{\Sigma_{\alpha'}\} \\ &\leq 2^j \varepsilon \sup_{\alpha' \in \mathbb{R}} \text{mes}\{\Sigma_{\alpha'}\} \leq C 2^j \varepsilon \end{aligned} \tag{68}$$

for a constant C that is independent of j , ε , and α . Hence, introducing a dyadic decomposition of \mathbb{T}^3 with respect to e_Δ centered about Σ_α , we find

$$\begin{aligned} \int_{\mathbb{T}^3} dp \frac{1}{|e_\Delta(p) - \alpha - i\varepsilon|} &\leq \sum_j \int_{R_j(\alpha, \varepsilon)} dp \left| \frac{1}{e_\Delta(p) - \alpha - i\varepsilon} \right| \\ &\leq C \sum_j \frac{\text{mes}\{R_j(\alpha, \varepsilon)\}}{2^j \varepsilon} \\ &\leq C \log \frac{1}{\varepsilon}, \end{aligned} \tag{69}$$

for $0 \leq j \leq C \log \frac{1}{\varepsilon}$ and a constant C that is uniform in ε and α , as claimed. ■

Lemma 3.5. Let

$$\begin{aligned} K^{(n)}(p_0, \dots, p_n; t) &:= \int ds_0 \dots ds_n \delta\left(t - \sum_{r=0}^n s_r\right) e^{-i \sum_{j=0}^n s_j e_\Delta(p_j)} \\ &= \frac{i e^{\varepsilon t}}{2\pi} \int_I d\alpha e^{-i\alpha t} \prod_{j=0}^n \frac{1}{e_\Delta(p_j) - \alpha - i\varepsilon}. \end{aligned}$$

Then there exists a finite constant C_μ for every $0 < \mu < 1$ such that

$$\|K^{(n)}(\cdot; t)\|_{L^2((\mathbb{T}^3)^{n+1})}^2 \leq \frac{(C_\mu t)^n}{(n!)^\mu}. \tag{70}$$

Proof. Clearly,

$$\begin{aligned} \|K^{(n)}(\cdot; t)\|_{L^\infty((\mathbb{T}^3)^{n+1})} &\leq \int_{\mathbb{R}_+^{n+1}} ds_0 \dots ds_n \delta(t - \sum s_r) = \frac{t^n}{n!}. \end{aligned}$$

Furthermore,

$$\begin{aligned}
 & \|K^{(n)}(\cdot; t)\|_{L^{2-\mu}((\mathbb{T}^3)^{n+1})}^{2-\mu} \\
 & \leq C \int_{(\mathbb{T}^3)^{n+2}} d\underline{p} \delta(p_n - p_{n+1}) \left[\int_I |d\alpha| \prod_{j=0}^n \frac{1}{|e_\Delta(p_j) - \alpha - i\varepsilon|} \right]^{2-\mu} \\
 & \leq C \int_{(\mathbb{T}^3)^{n+2}} d\underline{p} \delta(p_n - p_{n+1}) \int_I |d\alpha| \prod_{j=0}^n \frac{1}{|e_\Delta(p_j) - \alpha - i\varepsilon|^{2-\mu}} \\
 & \leq C_\mu^n t^{n(1-\mu)} \tag{71}
 \end{aligned}$$

for a finite constant C_μ . The claim then follows from interpolation. ■

We conclude that the ladder contribution can be estimated by

$$\begin{aligned}
 & \lambda^{2n} \int_{(\mathbb{T}^3)^{2n+2}} d\underline{p} \prod_{i=1}^n \delta(p_i - p_{2n+1-i}) \\
 & \quad \times K^{(n)}(p_0, \dots, p_n; t) \overline{K^{(n)}(p_{n+1}, \dots, p_{2n+1}; t)} \\
 & \leq \lambda^{2n} \|K^{(n)}(\dots; t)\|_{L^2((\mathbb{T}^3)^{n+1})}^2 \leq \frac{(C_\mu \lambda^2 t)^n}{(n!)^\mu} \tag{72}
 \end{aligned}$$

for $0 < \mu < 1$. It is clear that the product of delta distributions appearing here is equivalent to the one in (59).

3.4.2. Immediate Recollisions

We next estimate general simple pairings which include all possible combinations of type I and I' contractions. Given any type I or type I' delta function $\delta(p_i - p_{i-1} + p_j - p_{j-1})$ in a simple pairing graph, where $j > i$, one necessarily finds $i = j - 1$. Otherwise, either a crossing or a nesting pairing occurs. Hence, any type I or I' delta function in a simple pairing reduces to $\delta(p_{i+1} - p_{i-1})$, for some i .

Definition 3.8. A type I or type I' pairing of the form $\delta(p_{i+1} - p_{i-1})$ is called an immediate recollision.

The subintegral in $\text{Amp}[\pi]$ corresponding to an immediate recollision is given by either

$$\Xi(\alpha, \varepsilon) := \int_{\mathbb{T}^3} \frac{dq}{e_\Delta(q) - \alpha - i\varepsilon} \tag{73}$$

or $\Xi(\beta, -\varepsilon)$. It contributes to a renormalization of the particle propagator, see ref. 3, and satisfies the following estimates, which will be of extensive use.

Lemma 3.6. For $\alpha \in I$,

$$\sup_{\alpha \in I} \sup_{\varepsilon > 0} |\Xi(\alpha, \varepsilon)| < C \tag{74}$$

and

$$|\partial_\alpha^m \Xi(\alpha, \varepsilon)| \leq C \varepsilon^{-(m-1/2)} (m!) \tag{75}$$

$$|\Xi(\alpha, \varepsilon) - \Xi(\alpha', \varepsilon)| \leq C \varepsilon^{-1/2} |\alpha - \alpha'| \tag{76}$$

for finite constants C that are independent of $m, \varepsilon, \alpha, \alpha'$, and $m \in \mathbb{N}$.

Proof. We recall that $\alpha \in I = I_{\mathbb{R}} \cup I_{\mathbb{H}_-}$ from (51). The case $\alpha \in I_{\mathbb{H}_-}$ is trivial. For $\alpha \in I_{\mathbb{R}} = [-1, 7]$, we write

$$\Xi(\alpha, \varepsilon) = \int_{\mathbb{R}_+} ds \int_{\mathbb{T}^3} dp e^{-is(e_\Delta(p) - \alpha - i\varepsilon)},$$

and recall that $e_\Delta: \mathbb{T}^3 \rightarrow [0, 6]$ is a real analytic Morse function with eight critical points. We choose a smooth partition of unity $1 = \sum \phi_j$ on \mathbb{T}^3 , $j \in \{1, \dots, 8\}$, requiring that the support of each ϕ_j is centered about precisely one critical point of e_Δ , so that

$$\Xi(\alpha, \varepsilon) = \sum_j \int_{\mathbb{R}_+} ds \int_{\mathbb{T}^3} dp \phi_j(p) e^{-is(e_\Delta(p) - \alpha - i\varepsilon)}. \tag{77}$$

Using a stationary phase estimate, we find

$$|\Xi(\alpha, \varepsilon)| < \sum_j C_j \int_{\mathbb{R}_+} ds e^{-\varepsilon s} (1+s)^{-\frac{3}{2}}, \tag{78}$$

where the constants C_j are independent of ε and α . This proves (74).

Likewise,

$$\partial_\alpha^m \Xi(\alpha, \varepsilon) = \sum_j \int_{\mathbb{R}_+} ds \int_{\mathbb{T}^3} dp \phi_j(p) (is)^m e^{-is(e_\Delta(p) - \alpha - i\varepsilon)}. \tag{79}$$

Thus

$$|\partial_\alpha^m \Xi(\alpha, \varepsilon)| < \sum_j C_j \int_{\mathbb{R}^+} ds s^m e^{-\varepsilon s} (1+s)^{-\frac{3}{2}}, \tag{80}$$

which implies (75), and for $m = 1$, also (76). ■

3.4.3. General Simple Pairings

In a more general context, simple pairings comprise progressions of neighboring immediate recollisions on each particle line before and after each type II contraction. In this sense, simple pairings are ladder graphs that are decorated with immediate recollisions on the propagator lines. Let us assume that there are q neighboring delta functions of type I', starting at the particle propagator carrying the momentum p_i . Then, $\text{Amp}[\pi]$ contains the corresponding subintegral

$$\begin{aligned} \int_{(\mathbb{T}^3)^{2q}} dp_{i+1} \cdots dp_{i+2q} & \left(\prod_{l=i}^{i+2q} \frac{1}{e_\Delta(p_l) - \alpha - i\varepsilon} \right) \prod_{j=1}^q \delta(p_{i+2j+1} - p_{i+2j-1}) \\ & = \frac{\Xi(\alpha, \varepsilon)^q}{(e_\Delta(p_i) - \alpha - i\varepsilon)^{q+1}}. \end{aligned}$$

The analogous expression for a progression of q neighboring delta functions of type I is obtained from substituting $\alpha \rightarrow \beta$ and $\varepsilon \rightarrow -\varepsilon$.

Let us consider a simple pairing $\pi \in \Pi_{n,n'}$ which contains m type II contractions. Let

$$q^{(m+1)} := (q_0, q_1, \dots, q_m) \in \mathbb{N}_0^{m+1}$$

and

$$|q^{(m+1)}| := q_0 + \cdots + q_m,$$

and, for $n - n' \equiv 0 \pmod{2}$,

$$A_{n,n'} := \{m \in \mathbb{N}_0 \mid m - n \equiv 0 \pmod{2}, \quad m \leq \min\{n, n'\}\}. \tag{81}$$

The sum over all simple pairings at fixed n gives (after reindexing the momentum variables)

$$\begin{aligned}
 \sum_{\substack{\pi \in \Pi_{n,n'} \\ \pi \text{ simple}}} \text{Amp}[\pi] &= \sum_{m \in A_{n,n'}} \frac{\lambda^{2m} e^{2\varepsilon t}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\
 &\times \sum_{|q^{(m+1)}| = \frac{n-m}{2}} \sum_{|\tilde{q}^{(m+1)}| = \frac{n'-m}{2}} \int_{(\mathbb{T}^3)^{m+1}} dp_0 \cdots dp_m \\
 &\times \prod_{i=0}^m \frac{(\lambda^2 \Xi(\alpha, \varepsilon))^{q_i}}{(e_\Delta(p_i) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Xi(\beta, -\varepsilon))^{\tilde{q}_i}}{(e_\Delta(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}}.
 \end{aligned} \tag{82}$$

Let us comment on this expression, cf. Fig. 2. For $i = 1, \dots, m$, p_{i-1} is the momentum preceding, and p_i the momentum following the i th type II pairing. Notably, a direct recollision conserves the momentum. For $1 \leq i \leq m$, q_i and \tilde{q}_i are the numbers of neighboring type I and I' pairings after the i th type II contraction. q_0 and \tilde{q}_0 are the number of neighboring type I and I' pairings before the 1-st type II pairing.

Clearly, all $n - m$ random potentials on each particle line not involved in type II contractions are part of type I, respectively type I' pairings (immediate recollisions). Since each immediate recollision contracts precisely two random potentials, the sum over m takes steps of size 2, such that $m \in A_{n,n'}$. Therefore,

$$|q^{(m+1)}| = \frac{n-m}{2}, \quad |\tilde{q}^{(m+1)}| = \frac{n'-m}{2} \tag{83}$$

is clear. In particular, $m = n = n'$ corresponds to the ladder graph.

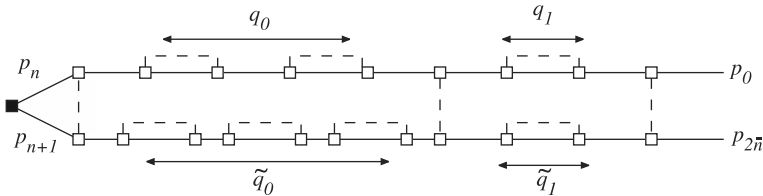


Fig. 2. A simple pairing contraction graph.

Lemma 3.7. For fixed n, n' with $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$, the contribution of the sum of all simple pairings is bounded by

$$\begin{aligned} & \left| \sum_{\pi \in \Pi_{n,n'} \text{ simple}} \text{Amp}[\pi] \right| \\ & \leq \delta_{n,n'} \frac{(C_0 \lambda^2 \varepsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{1/2}} + \bar{n}^2 \varepsilon^{\frac{1}{2}} |\log \varepsilon| (C \varepsilon^{-1} \lambda^2 |\log \varepsilon|)^{\bar{n}}, \end{aligned} \tag{84}$$

and for $\lambda^2 \varepsilon^{-1} \leq 1$,

$$\begin{aligned} & \sum_{n,n'=1}^N \left| \sum_{\pi \in \Pi_{n,n'} \text{ simple}} \text{Amp}[\pi] \right| \\ & \leq C_1 \lambda^2 \varepsilon^{-1} + \varepsilon^{\frac{1}{2}} N^3 (C \lambda^2 \varepsilon^{-1} |\log \varepsilon|)^N, \end{aligned} \tag{85}$$

where C, C_0, C_1 are uniform in N and ε , and where C_0 and C_1 are defined in (96).

Proof. Let us assume for fixed n, n' under the stated conditions that $\pi \in \Pi_{n,n'}$ is simple, and contains m type II pairings. Let

$$\text{Amp}[\pi] = \text{Amp}_{\text{main}}[\pi] + \text{Amp}_{\text{error}}[\pi], \tag{86}$$

where

$$\begin{aligned} \text{Amp}_{\text{main}}[\pi] & := \frac{e^{2\varepsilon t} \lambda^{2m}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ & \times \sum_{|q^{(m+1)}| = \frac{n-m}{2}} \sum_{|\tilde{q}^{(m+1)}| = \frac{n'-m}{2}} \int_{(\mathbb{T}^3)^{m+1}} dp_0 \cdots dp_m \\ & \times \prod_{i=0}^m \frac{(\lambda^2 \Xi(e_\Delta(p_0), \varepsilon))^{q_i}}{(e_\Delta(p_i) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Xi(e_\Delta(p_0), -\varepsilon))^{\tilde{q}_i}}{(e_\Delta(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}}. \end{aligned} \tag{87}$$

Then, recalling (81),

$$\begin{aligned}
 \sum_{\pi \in \Pi_{n,n'} \text{ simple}} \text{Amp}_{\text{main}}[\pi] &= \sum_{m \in A_{n,n'}} \frac{e^{2\varepsilon t} \lambda^{2m}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\
 &\times \sum_{|q^{(m+1)}| = \frac{n-m}{2}} \sum_{|\tilde{q}^{(m+1)}| = \frac{n'-m}{2}} \int_{(\mathbb{T}^3)^{m+1}} dp_0 \cdots dp_m \\
 &\times \prod_{i=0}^m \frac{(\lambda^2 \Xi(e_\Delta(p_0), \varepsilon))^{q_i}}{(e_\Delta(p_i) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Xi(e_\Delta(p_0), -\varepsilon))^{\tilde{q}_i}}{(e_\Delta(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}}.
 \end{aligned} \tag{88}$$

Let $p_j^{(q_j)} = (p_j, \dots, p_j)$ (q_j copies), and $d\underline{p}^{(m+1)} := dp_0 \cdots dp_m$.
 We note that

$$\int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} \left| K^{(\frac{m+n}{2})}(p_0^{(q_0+1)}, \dots, p_m^{(q_m+1)}; t) \right|^2 \leq \frac{(C\mu t)^{\frac{m+n}{2}}}{(\frac{m+n}{2}!)^{\frac{1}{2}}}. \tag{89}$$

Thus, by the Schwarz inequality,

$$\begin{aligned}
 &\int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} K^{(\frac{n+m}{2})}(p_0^{q_0+1}, \dots, p_m^{q_m+1}; t) \\
 &\quad \times \overline{K^{(\frac{n'+m}{2})}(p_0^{\tilde{q}_0+1}, \dots, p_m^{\tilde{q}_m+1}; t)} \\
 &\leq \frac{(Ct)^{\frac{m+n}{2}}}{(\frac{m+n}{2}!)^{\frac{1}{4}} (\frac{m+n'}{2}!)^{\frac{1}{4}}}.
 \end{aligned} \tag{90}$$

Therefore,

$$|(88)| \leq \sum_{m \in A_{n,n'}} \sum_{|q^{(m+1)}| = \frac{n-m}{2}} \sum_{|\tilde{q}^{(m+1)}| = \frac{n'-m}{2}} \frac{2\lambda^{2m} (Ct)^{\frac{m+n}{2}} |\lambda^2 \Xi(p_0; \varepsilon)|^{\tilde{n}-m}}{(\frac{m+n}{2}!)^{\frac{1}{4}} (\frac{m+n'}{2}!)^{\frac{1}{4}}}. \tag{91}$$

By

$$\sum_{|q^{(m+1)}| = \frac{n-m}{2}} 1 < C^{n-m} \tag{92}$$

and Lemma 3.6,

$$\begin{aligned} \left| \sum_{\pi \in \Pi_{n,n} \text{ simple}} \text{Amp}_{\text{main}}[\pi] \right| &\leq \sum_{m \in A_{n,n'}} \frac{\lambda^{2m} (C\lambda^2)^{\bar{n}-m} (Ct)^{\frac{m+\bar{n}}{2}}}{\left(\frac{m+n}{2}\right)^{\frac{1}{4}} \left(\frac{m+n'}{2}\right)^{\frac{1}{4}}} \\ &\leq \delta_{n,n'} \frac{(C\lambda^2 t)^{\bar{n}}}{(\bar{n}!)^{\frac{1}{2}}} + C\bar{n}t^{-1} (C\lambda^2 t)^{\bar{n}} \end{aligned} \tag{93}$$

for finite constants C that are independent of ε . The first term after the second inequality sign accounts for the ladder graph in $\Gamma_{n,n}$, corresponding to the case $m = n = n' = \bar{n}$, as we recall.

Next, we consider the error term

$$\begin{aligned} \sum_{\pi \in \Pi_{n,n} \text{ simple}} \text{Amp}_{\text{error}}[\pi] &= \frac{1}{(2\pi)^2} \int_{\mathbb{I} \times \bar{\mathbb{I}}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ &\times \sum_{m \in A_{n,n'}} \sum_{|q^{(m+1)}| = \frac{n-m}{2}} \sum_{|\tilde{q}^{(m+1)}| = \frac{n'-m}{2}} \lambda^{2m+2\sum_{j=0}^m (q_j + \tilde{q}_j)} \\ &\times \int_{(\mathbb{T}^3)^{m+1}} d\underline{p}^{(m+1)} \prod_{i=0}^m \frac{1}{(e_{\Delta}(p_i) - \alpha - i\varepsilon)^{q_i+1} (e_{\Delta}(p_i) - \beta + i\varepsilon)^{\tilde{q}_i+1}} \\ &\times \left[\Xi(\alpha, \varepsilon)^{\frac{n-m}{2}} \Xi(\beta, -\varepsilon)^{\frac{n'-m}{2}} - \Xi(e_{\Delta}(p_0), \varepsilon)^{\frac{n-m}{2}} \Xi(e_{\Delta}(p_0), -\varepsilon)^{\frac{n'-m}{2}} \right]. \end{aligned} \tag{94}$$

Lemma 3.6 implies that difference in $[\dots]$ on the last line is bounded by

$$\left[\frac{n-m}{2} |e_{\Delta}(p_0) - \alpha| + \frac{n'-m}{2} |e_{\Delta}(p_0) - \beta| \right] \varepsilon^{-1/2} C^{\bar{n}-m-1},$$

for a constant C independent of ε . Thus, we arrive at

$$\begin{aligned} |(94)| &\leq \bar{n} \varepsilon^{1/2} \sum_{m \in A_{n,n'}} (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^m (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}-m} \\ &\leq \bar{n}^2 \varepsilon^{1/2} (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}}. \end{aligned}$$

again using (92).

Summarizing, we have

$$\begin{aligned} \left| \sum_{\pi \in \Pi_{n,n'} \text{ simple}} \text{Amp}[\pi] \right| &\leq \delta_{n,n'} \frac{(C_0 \lambda^2 \varepsilon^{-1})^{\bar{n}}}{(\bar{n}!)^{\frac{1}{2}}} + n\varepsilon (C\lambda^2 \varepsilon^{-1})^{\bar{n}} \\ &\quad + \bar{n}^2 \varepsilon^{1/2} (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}} \end{aligned} \tag{95}$$

for some constant C_0 . Furthermore, let

$$C_1 := \sum_{\bar{n}=1}^{\infty} \frac{C_0^{\bar{n}}}{(\bar{n}!)^{\frac{1}{2}}}. \tag{96}$$

Then, for $\lambda^2 \varepsilon^{-1} \leq 1$,

$$\sum_{n=1}^N |(82)| \leq C_1 \lambda^2 \varepsilon^{-1} + \varepsilon^{1/2} N^3 (C \lambda^2 \varepsilon^{-1} |\log \varepsilon|)^N,$$

where the constants C_0, C_1, C are uniform in N, λ, ε . ■

We remark that for general $T := \lambda^2 t = \lambda^2 \varepsilon^{-1} > 1$, the constant C_1 in the above estimate would be replaced by C^T , where C is uniform in λ and $\varepsilon = t^{-1}$.

3.4.4. A priori Bound on Pairing Graphs

All pairing graphs obey the following *a priori* bound.

Lemma 3.8. Let $\pi \in \Pi_{n,n'}$ be a pairing graph, and $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$. Then,

$$|\text{Amp}[\pi]| \leq |\log \varepsilon|^3 (C \lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}}. \tag{97}$$

Proof. For the detailed argument, we refer the reader to ref. 3. One chooses a complete spanning tree T on π , and estimates the propagators supported on T in L^∞ . Using the bounds in Lemma 3.4, one obtains a factor $\varepsilon^{-\bar{n}}$. The loop propagators are estimated in L^1 , and yield a factor $(C |\log \varepsilon|)^{\bar{n}+3}$. ■

3.4.5. Crossing and Nested Pairings

We shall next prove that for all $\pi \in \Pi_{n,n'}$ which contain a crossing or nested pairing contraction, $|\text{Amp}[\pi]|$ is a factor $O(\varepsilon^{\frac{1}{5}})$ smaller than the *a priori* bound (97) on pairing graphs. This is sufficient to compensate the factor $\bar{n}!$ accounting for the number of pairing contractions.

Lemma 3.9. The sum of all crossing and nested pairing contractions in $\Pi_{n,n'}$ (where $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$) is bounded by

$$\sum_{\pi \in \Pi_{n,n'} \text{ crossing or nested}} |\text{Amp}[\pi]| \leq \bar{n}! \varepsilon^{\frac{1}{2}} |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}}.$$

Proof. By lemmata 3.10 and 3.12 below, every pairing contraction of crossing or nesting type can be bounded by

$$(C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}} \varepsilon^{\frac{1}{2}} |\log \varepsilon|^3,$$

and clearly, there are at most $2^{\bar{n}} \bar{n}!$ such graphs. ■

Lemma 3.10. Suppose that $\pi \in \Pi_{n,n'}$ corresponds to a pairing contraction that contains at least one crossing, and that $\bar{n} = \frac{n+n'}{2} \in \mathbb{N}$. Then,

$$|\text{Amp}[\pi]| \leq \varepsilon^{\frac{1}{2}} (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}} |\log \varepsilon|^3.$$

Proof. Let T denote a complete spanning tree for the graph G_π , and T^c its complement. As demonstrated in ref. 3, all momenta supported on T can be expressed as linear combinations of loop momenta supported on T^c . If there exists a crossing pairing, it is shown in ref. 3 that there is a tree momentum p_r in T that depends on at least two loop momenta p_j, p_l in T^c ,

$$p_r = \pm p_j \pm p_l \pm w, \tag{98}$$

where $w \in \mathbb{T}^3$ is a linear combination of momenta not depending on p_j, p_l . Writing $p \equiv p_j, q \equiv p_l$, and integrating out all delta distributions determined by π against momenta supported on T , the amplitude $\text{Amp}[\pi]$ can be written in the form

$$\begin{aligned} \text{Amp}[\pi] &= \frac{e^{2\varepsilon i} \lambda^{2\bar{n}}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \\ &\times \int_{\mathbb{T}^{3|T^c|}} dp dq \left[\prod_{\substack{p_j \in T^c \\ p_j \neq p, q}} dp_j \right] F_\pi(p_j \in T^c; \alpha, \beta; \varepsilon) \\ &\times \frac{1}{(e_\Delta(p) - \alpha_1 \pm i\varepsilon)(e_\Delta(q) - \alpha_2 \pm i\varepsilon)(e_\Delta(p \pm q + w) - \alpha_3 \pm i\varepsilon)}, \end{aligned} \tag{99}$$

where $\alpha_i \in \{\alpha, \beta\}$, for $i = 1, 2, 3$, and $|T^c|$ is the number of edges of T^c . F_π contains all resolvents except the three on the last line, which carry the moments singled out in (98). Using an $L^1 - L^\infty$ bound with respect to the variables p, q and α, β , we have

$$\begin{aligned}
 |\text{Amp}[\pi]| &\leq \lambda^{2\bar{n}} \sup_{\alpha \in I} \sup_{\beta \in \bar{I}} \sup_{w \in \mathbb{T}^3} A_\varepsilon(w; \alpha, \beta) \\
 &\times \sup_{p, q} \int_{I \times \bar{I}} d\alpha d\beta \left| \int_{\mathbb{T}^3(|T^c|-2)} \left[\prod_{\substack{p_j \in T^c \\ p_j \neq p, q}} dp_j \right] F_\pi(p_j \in T^c; \alpha, \beta; \varepsilon) \right|,
 \end{aligned}
 \tag{100}$$

where

$$\begin{aligned}
 A_\varepsilon(w; \alpha, \beta) &:= \int_{(\mathbb{T}^3)^2} \frac{dp dq}{|e_\Delta(p) - \alpha_1 \pm i\varepsilon| |e_\Delta(q) - \alpha_2 \pm i\varepsilon| |e_\Delta(p \pm q + w) - \alpha_3 \pm i\varepsilon|}.
 \end{aligned}
 \tag{101}$$

It is clear that $\alpha_i = \alpha_j$ for at least one pair of indices $i \neq j$.

Using the trivial bound $A_\varepsilon(w, \alpha, \beta) \leq c\varepsilon^{-1} (\log \frac{1}{\varepsilon})^2$, one obtains

$$|\text{Amp}[\pi]| \leq |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}},
 \tag{102}$$

which is the *a priori* bound (97) on all pairing graphs. It is insufficient because the number of crossing graphs is $O(\bar{n}!)$, and $\bar{n}! |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}}$ is not summable in \bar{n} . Gaining an extra factor $\varepsilon^{\frac{1}{5}}$ will (in combination with our treatment of the error term of the truncated Duhamel expansion) allow us to compensate the large combinatorial factor $\bar{n}!$.

Exploiting the crossing structure of π , Lemma 3.11 below provides the bound

$$\sup_{w \in \mathbb{T}^3} \sup_{\alpha \in I} \sup_{\beta \in \bar{I}} A_\varepsilon(w, \alpha, \beta) < C \varepsilon^{-\frac{4}{5}} \left(\log \frac{1}{\varepsilon} \right)^2,
 \tag{103}$$

which is a factor $\varepsilon^{\frac{1}{5}}$ smaller than the *a priori* estimate.

For the remaining part of $\text{Amp}[\pi]$, excluding the propagators corresponding to the indices n and $n + 1$, L^∞ -bounds on propagators in T , and L^1 -bounds on propagators in T^c , produce a factor $(C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}-1}$. The propagators corresponding to the indices n and $n + 1$ contribute a factor $(C \log \varepsilon^{-1})^2$, as in (134) below. A detailed exposition is given in refs. 2, 3. ■

Lemma 3.11. Let $A_\varepsilon(w, \alpha, \beta)$ be defined as in (101). Then,

$$\sup_{w \in \mathbb{T}^3} \sup_{\alpha \in I} \sup_{\beta \in \bar{I}} A_\varepsilon(w, \alpha, \beta) < C\varepsilon^{-4/5} \left(\log \frac{1}{\varepsilon} \right)^2. \tag{104}$$

Proof. To bound (101), it is necessary to estimate the measure of the intersection between tubular neighborhoods of level surfaces of the kinetic energy function e_Δ where the singularities of the resolvents in (101) are concentrated. Because the level surfaces of e_Δ are non-convex for the three-dimensional lattice model, this is a much more difficult task than in the continuum case, where the latter are spheres. After completing this work, we learned that a similar but somewhat stronger estimate (with an exponent $-3/4$ instead of $-4/5$) was proven independently in ref. 4.

We shall interpret the three-dimensional integral (101) as an average over two-dimensional crossing integrals. Let

$$\begin{aligned} \underline{p} &:= (p_1, p_2), \quad \underline{q} := (q_1, q_2) \in [0, 1]^2 \\ e_{2D}(\underline{p}) &:= \sum_{j=1}^2 \cos 2\pi p_j \\ \alpha_j(k) &:= -\cos 2\pi k + 3 - \begin{cases} \beta & \text{if } j=1 \\ \alpha & \text{if } j=2, 3, \end{cases} \end{aligned} \tag{105}$$

so that

$$\begin{aligned} A_\varepsilon(w; \alpha, \beta) & \\ &= \int_{[0,1]} dp_3 \int_{[0,1]} dq_3 \int_{[0,1]^2} dq \frac{1}{|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3) + i\varepsilon|} \\ &\quad \times \int_{[0,1]^2} d\underline{p} \frac{1}{|e_{2D}(\underline{p}) - \alpha_2(p_3) - i\varepsilon| |e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3) - i\varepsilon|}. \end{aligned} \tag{106}$$

The level curves

$$s_\alpha := \{ \underline{p} \in [0, 1]^2 \mid e_{2D}(\underline{p}) = \alpha \} \tag{107}$$

of the two-dimensional kinetic energy function e_{2D} are convex, but there is one exceptional value of the energy $\alpha=0$ for which the corresponding level curve $s_{\alpha=0}$ is the union of four line segments of zero curvature. The lack of curvature poses a well-known difficulty in two-dimensional lattice models. In three-dimensions, this problem is resolved through the average with

respect to p_3, q_3 (relative to which small curvature is an event of small probability).

Let

$$\begin{aligned} U_\tau &:= \{(p_3, q_3) \mid \alpha_1(q_3), \alpha_2(p_3), \alpha_3(p_3 - q_3) \notin (-\tau, \tau)\} \\ U_\tau^c &:= [0, 1]^2 \setminus U_\tau, \end{aligned} \tag{108}$$

where $0 < \tau \ll 1$ remains to be optimized. Then, clearly,

$$\text{mes}\{U_\tau^c\} < C\sqrt{\tau}. \tag{109}$$

Correspondingly, let

$$A_\varepsilon(w; \alpha, \beta) = (A) + (B), \tag{110}$$

where

$$\begin{aligned} (A) &:= \int_{U_\tau} dp_3 dq_3 \int_{[0,1]^2} dq \frac{1}{|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3) + i\varepsilon|} \\ &\quad \times \int_{[0,1]^2} dp \frac{1}{|e_{2D}(\underline{p}) - \alpha_2(p_3) - i\varepsilon| |e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3) - i\varepsilon|}, \end{aligned} \tag{111}$$

and

$$\begin{aligned} (B) &:= \int_{U_\tau^c} dp_3 dq_3 \int_{[0,1]^2} dq \frac{1}{|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3) + i\varepsilon|} \\ &\quad \times \int_{[0,1]^2} dp \frac{1}{|e_{2D}(\underline{p}) - \alpha_2(p_3) - i\varepsilon| |e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3) - i\varepsilon|}. \end{aligned} \tag{112}$$

Therefore, with (109),

$$\begin{aligned} (B) &< C\sqrt{\tau} \sup_{\alpha_j} \int_{[0,1]^2} dq \frac{1}{|e_{2D}(\underline{q} - \underline{w}) - \alpha_1 + i\varepsilon|} \\ &\quad \times \int_{[0,1]^2} dp \frac{1}{|e_{2D}(\underline{p}) - \alpha_2 - i\varepsilon| |e_{2D}(\underline{p} - \underline{q}) - \alpha_3 - i\varepsilon|} \\ &< \frac{C\sqrt{\tau}}{\varepsilon} \left(\log \frac{1}{\varepsilon}\right)^2. \end{aligned} \tag{113}$$

Next, we decompose (A) into $(A) = (A_1) + (A_2)$ with

$$\begin{aligned}
 (A_1) &:= \int_{U_\tau} dp_3 dq_3 \int_{[0,1]^2} d\underline{q} \frac{\chi(|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3)| < \eta)}{|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3) + i\varepsilon|} \\
 &\quad \times \int_{[0,1]^2} d\underline{p} \frac{\chi(|e_{2D}(\underline{p}) - \alpha_2(p_3)| < \eta)}{|e_{2D}(\underline{p}) - \alpha_2(p_3) - i\varepsilon|} \\
 &\quad \times \frac{\chi(|e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3)| < \eta)}{|e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3) - i\varepsilon|}, \tag{114}
 \end{aligned}$$

where $0 < \eta \ll 1$ remains to be determined. Then clearly, the term complementary to (A_1) satisfies

$$(A_2) < C\eta^{-1} \left(\log \frac{1}{\varepsilon} \right)^2, \tag{115}$$

because the integrand of (A_2) contains at least one characteristic function of the form $\chi(|e_{2D}(\underline{v}) - \alpha_j(v_3)| > \eta)$, where $v = (\underline{v}, v_3)$ denotes either $q - w$, p , or $p - q$. The corresponding resolvent can be estimated by η^{-1} , while the remaining two resolvents in (A_2) can be bounded in L^1 by $c(\log \frac{1}{\varepsilon})^2$.

To bound (A_1) , we note that

$$\begin{aligned}
 (A_1) &< \left(\frac{\eta}{\varepsilon} \right)^3 \eta^{-3} \int_{U_\tau} dp_3 dq_3 \int_{[0,1]^2} d\underline{q} \chi(|e_{2D}(\underline{q} - \underline{w}) - \alpha_1(q_3 - w_3)| < \eta) \\
 &\quad \times \int_{[0,1]^2} d\underline{p} \chi(|e_{2D}(\underline{p}) - \alpha_2(p_3)| < \eta) \chi(|e_{2D}(\underline{p} - \underline{q}) - \alpha_3(p_3 - q_3)| < \eta). \tag{116}
 \end{aligned}$$

let h_η denote a smooth bump function supported in a η -vicinity of the origin in $[0, 1]^2$ (periodically continued over the boundaries), with

$$\|h_\eta\|_{L^1([0,1]^2)} < 10 \tag{117}$$

and

$$|\mathcal{F}^{-1}(h_\eta)(\underline{x})| \begin{cases} \approx C & |\underline{x}| < \eta^{-1} \\ < C|\underline{x}|^{-1} & |\underline{x}| \geq \eta^{-1}. \end{cases} \tag{118}$$

\mathcal{F} denotes the Fourier transform, and $\underline{x} \in \mathbb{Z}^2$ is the variable conjugate to \underline{p} , respectively \underline{q} . Furthermore, let

$$\delta_{s_\alpha}^{(\eta)}(\underline{p}) := h_\eta * \delta_{s_\alpha}(\underline{p}), \tag{119}$$

where $\delta_{s_\alpha}(\underline{p}) := \delta(e_{2D}(\underline{p}) - \alpha)$.

Choosing h_η appropriately,

$$\frac{1}{\eta} \chi(|e_{2D}(\underline{p}) - \alpha| < \eta) < \delta_{s_\alpha}^{(\eta)}(\underline{p}). \quad (120)$$

Thus, letting $T_{\underline{w}}f(\underline{q}) := f(\underline{q} - \underline{w})$,

$$\begin{aligned} (A_1) &\leq \left(\frac{\eta}{\varepsilon}\right)^3 \sup_{|\alpha_j| > \tau} \sup_{\underline{w}} \int_{[0,1]^2} d\underline{q} \delta_{s_{\alpha_1}}^{(\eta)}(\underline{q} - \underline{w}) \int_{[0,1]^2} d\underline{p} \delta_{s_{\alpha_2}}^{(\eta)}(\underline{p}) \delta_{s_{\alpha_3}}^{(\eta)}(\underline{p} - \underline{q}) \\ &= \left(\frac{\eta}{\varepsilon}\right)^3 \sup_{|\alpha_j| > \tau} \sup_{\underline{w}} \langle T_{\underline{w}}\delta_{s_{\alpha_1}}^{(\eta)}, \delta_{s_{\alpha_2}}^{(\eta)} * \delta_{s_{\alpha_3}}^{(\eta)} \rangle_{L^2([0,1]^2)} \\ &= \left(\frac{\eta}{\varepsilon}\right)^3 \sup_{|\alpha_j| > \tau} \sup_{\underline{w}} \langle \mathcal{F}^{-1}(T_{\underline{w}}\delta_{s_{\alpha_1}}^{(\eta)}), \mathcal{F}^{-1}(\delta_{s_{\alpha_2}}^{(\eta)}) \mathcal{F}^{-1}(\delta_{s_{\alpha_3}}^{(\eta)}) \rangle_{\ell^2(\mathbb{Z}^2)} \\ &= \left(\frac{\eta}{\varepsilon}\right)^3 \sup_{|\alpha_j| > \tau} \sup_{\underline{w}} \sum_{\underline{x} \in \mathbb{Z}^2} \mathcal{F}^{-1}(T_{\underline{w}}\delta_{s_{\alpha_1}}^{(\eta)})(\underline{x}) \mathcal{F}^{-1}(\delta_{s_{\alpha_2}}^{(\eta)})(\underline{x}) \mathcal{F}^{-1}(\delta_{s_{\alpha_3}}^{(\eta)})(\underline{x}) \\ &= \left(\frac{\eta}{\varepsilon}\right)^3 \sup_{|\alpha_j| > \tau} \sup_{\underline{w}} \sum_{\underline{x} \in \mathbb{Z}^2} (\mathcal{F}^{-1}(h_\eta)(\underline{x}))^3 \mathcal{F}^{-1}(T_{\underline{w}}\delta_{s_{\alpha_1}})(\underline{x}) \\ &\quad \times \mathcal{F}^{-1}(\delta_{s_{\alpha_2}})(\underline{x}) \mathcal{F}^{-1}(\delta_{s_{\alpha_3}})(\underline{x}), \end{aligned} \quad (121)$$

by the Plancherel identity. Next, we observe that if $|\alpha| > \tau$, the curvature of $s_\alpha \subset [0, 1]^2$ is uniformly bounded below by $C\tau$, where the constant C is independent of τ . We thus have the curvature induced decay estimate

$$|\mathcal{F}^{-1}(\delta_{s_\alpha})(\underline{x})| < C(\tau|\underline{x}|)^{-\frac{1}{2}}, \quad (122)$$

which appears also in the context of restriction estimates in harmonic analysis.⁽¹⁰⁾ To arrive at (122), one introduces a smooth partition of unity $1 = \sum_{j=1}^N g_j$ on s_α , which splits it into N arcs $s_{\alpha,j}$, $j = 1, \dots, N$, with, say, $N = 10$. For fixed j , one introduces a local orthogonal coordinate system (v_1, v_2) where the origin lies on $s_{\alpha,j}$ (say at its center) with the v_1 -axis tangent to $s_{\alpha,j}$. Then, $s_{\alpha,j}$ is the graph of a smooth function $\phi_{\alpha,j}(v_1)$ with $\phi_{\alpha,j}(0) = 0$, $\partial_{v_1}\phi_{\alpha,j}(v_1) = 0$ and $|\partial_{v_1}^2\phi_{\alpha,j}(v_1)| > C\tau$. Let $\underline{n} := \frac{\underline{x}}{|\underline{x}|} = (n_1, n_2)$, where $\underline{x} \in \mathbb{Z}^2$, and let \underline{p}_j denote the location of the origin of the \underline{v} -coordinate system with respect to the \underline{p} -coordinates. Then,

$$\begin{aligned} &\mathcal{F}^{-1}(\delta_{s_{\alpha,j}}g_j)(\underline{x}) \\ &= e^{2\pi i \underline{x} \underline{p}_j} \int dv_1 (1 + (\partial_{v_1}\phi_{\alpha,j}(v_1))^2)^{\frac{1}{2}} \tilde{g}_j(v_1) e^{2\pi i |\underline{x}| \Phi_{\alpha,j}(\underline{n}, v_1)}, \end{aligned}$$

where $\tilde{g}_j(v_1) := g_j(v_1, \phi_{\alpha,j}(v_1))$, and

$$\Phi_{\alpha,j}(\underline{n}, v_1) := n_1 v_1 + n_2 \phi_{\alpha,j}(v_1). \tag{123}$$

First of all, if \underline{x} is parallel to the v_2 -axis so that $n_1 = 0$, one has $|\partial_{v_1} \Phi_{\alpha,j}(\underline{n}, 0)| = 0$ and

$$|\partial_{v_1}^2 \Phi_{\alpha,j}(\underline{n}, 0)| > C\tau. \tag{124}$$

Hence by a stationary phase estimate,

$$|\mathcal{F}^{-1}(\delta_{s_{\alpha,j}} g_j)(\underline{x})| < C(\tau|\underline{x}|)^{-\frac{1}{2}}. \tag{125}$$

If \underline{x} is close to being parallel to the v_2 -axis, so that $|n_1| < C$ is sufficiently small (independently of τ), one can find $v_1 = v_1(\underline{n})$, such that $\partial_{v_1} \Phi_{\alpha,j}(\underline{n}, v_1(\underline{n})) = 0$. This follows from an application of the implicit function theorem and (124). Moreover, by the assumption on the curvature of s_α , we have $|\partial_{v_1}^2 \Phi_{\alpha,j}(\underline{n}, v_1(\underline{n}))| > C\tau$. Therefore, (125) is valid for all \underline{x} such that $|n_1| < C$ is sufficiently small. If $|n_1| > C$,

$$|\partial_{v_1} \Phi_{\alpha,j}(\underline{n}, v_1)| > C' > 0, \tag{126}$$

since $|\partial_{v_1} \phi_{\alpha,j}(v_1)| = O(|v_1|)$. This implies an even stronger decay bound than (125), by standard oscillatory integral estimates. Hence, we arrive at (122).

Noting that

$$|\mathcal{F}^{-1}(T_w \delta_{s_\alpha})(\underline{x})| = |\mathcal{F}^{-1}(\delta_{s_\alpha})(\underline{x})|, \tag{127}$$

we obtain

$$\begin{aligned} (A_1) &< C \left(\frac{\eta}{\varepsilon}\right)^3 \sum_{\underline{x} \in \mathbb{Z}^2} (\mathcal{F}^{-1}(h_\eta)(\underline{x}))^3 (\tau|\underline{x}|)^{-\frac{3}{2}} \\ &< C \left(\frac{\eta}{\varepsilon}\right)^3 \tau^{-\frac{3}{2}} \eta^{\frac{1}{2}}, \end{aligned} \tag{128}$$

due to (118).

We thus arrive at

$$A_\varepsilon(w; \alpha, \beta) < C \left(\frac{\eta}{\varepsilon}\right)^3 \tau^{-\frac{3}{2}} \eta^{\frac{1}{2}} + C \left(\eta^{-1} + \frac{\sqrt{\tau}}{\varepsilon}\right) \left(\log \frac{1}{\varepsilon}\right)^2. \tag{129}$$

Setting $\eta = \varepsilon^{\frac{4}{3}}$, $\tau = \varepsilon^{\frac{2}{3}}$, the claim follows. ■

Lemma 3.12. Let $\pi \in \Pi_{n,n'}$ with $\bar{n} = \frac{n+n'}{2} \in \mathbb{N}$, correspond to a non-crossing pairing contraction that contains at least one nested subgraph. Then,

$$|\text{Amp}[\pi]| \leq \varepsilon^{\frac{1}{2}} (C\lambda^2\varepsilon^{-1} |\log \varepsilon|)^{\bar{n}}$$

Proof. In this case, π comprises a nested subgraph of length $1 < q \leq n - 2$, and $\text{Amp}[\pi]$ thus contains a subintegral

$$\begin{aligned} N_q(\alpha, \varepsilon)\delta(p_{j+2q} - p_j) &= \int_{(\mathbb{T}^3)^{2q}} dp_j \cdots dp_{j+2q-1} \prod_{l=j+1}^{j+2q-2} \frac{1}{e_{\Delta}(p_l) - \alpha - i\varepsilon} \\ &\quad \times \prod_{k=1}^{q-1} \lambda^2 \delta(p_{j+2k+2} - p_{j+2k}) \\ &\quad \times \lambda^2 \delta(p_{j+1} - p_j + p_{j+2q} - p_{j+2q-1}), \end{aligned} \tag{130}$$

where

$$N_q(\alpha, \varepsilon) := \lambda^2 \int_{\mathbb{T}^3} dp \frac{(\lambda^2 \Xi(\alpha, \varepsilon))^{q-1}}{(e_{\Delta}(p) - \alpha - i\varepsilon)^q}. \tag{131}$$

Since its interior does not contain further nested subgraphs, we refer to it as a simple nest, cf. Fig. 3. We note that p_0, p_n, p_{n+1} , and $p_{2\bar{n}+1}$ can never appear in the interior of a nested subgraph. It is clear that

$$\begin{aligned} |N_q(\alpha, \varepsilon)| &\leq \lambda^2 |\lambda^2 \Xi(\alpha, \varepsilon)|^{q-1} \left| \frac{1}{(q-1)!} \partial_{\alpha}^{q-1} \int_{\mathbb{T}^3} dp \frac{1}{e_{\Delta}(p) - \alpha - i\varepsilon} \right| \\ &\leq \frac{\lambda^2 (c\lambda^2)^{q-1}}{(q-1)!} \int_{\mathbb{R}_+} ds s^{q-1} e^{-\varepsilon s} (1+s)^{-\frac{3}{2}} \\ &\leq (C\lambda^2\varepsilon^{-1})^q \varepsilon^{\frac{3}{2}}, \end{aligned} \tag{132}$$

where we have used (75).

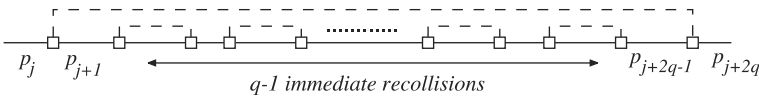


Fig. 3. A simple nest.

Without any loss of generality, let us assume that the simple nest has length q (i.e. it contains q immediate recollisions), and that the momentum with largest label preceding it is p_j , with $j + 2q < n$, so that the expression corresponding to (130) is $N_q \delta(p_{j+2q} - p_j)$.

The contributions to $\text{Amp}[\pi]$ stemming from the pairing contractions outside of the simple nest can be estimated in the following way. There are $2(\bar{n} - q)$ momenta not contained in the nest, apart from those carrying indices in $J := \{n, n + 1, j + 2q\}$. Let π' denote the graph obtained from π by removing the simple nest together with the edges labelled by J . Let T denote a spanning tree of π' containing all of the contraction lines, and $\bar{n} - q$ of the particle lines in π' . The pairings supported on π' can be written in the form

$$\int_{(\mathbb{T}^3)^{\bar{n}-q}} \left[\prod_{r \in T^c} \frac{dp_r}{(e_\Delta(p_r) - \alpha_r - i\sigma_r \varepsilon)^{\mu_j(r)}} \right] \prod_{s \in T} \frac{1}{(e_\Delta(w_s) - \alpha_s - i\sigma_s \varepsilon)^{\mu_j(s)}},$$

where each $w_s \in \mathbb{T}^3$ is a linear combination of momenta p_{j_s} with $j_s \in T^c$. Here, we have introduced $\mu_j(r) := 1 + \delta_{j,r}$ to accommodate the fact that the edge labelled by $j + 2q$ is excluded from π' . We correct this omission by using $p_{j+2q} = p_j$, which is enforced by a delta distribution, and by squaring the propagator corresponding to the edge with index j .

It then follows that $|\text{Amp}[\pi]|$ is bounded by

$$\begin{aligned} & \left(\sup_{\alpha \in I} |N_q(\alpha, \varepsilon)| \right) \lambda^{2(n-q)} \int_{I \times \bar{I}} |d\alpha| |d\beta| \\ & \times \int_{\mathbb{T}^3} dp_n \frac{1}{|e_\Delta(p_n) - \alpha - i\varepsilon|^{\mu_j(n)} |e_\Delta(p_n) - \beta + i\varepsilon|} \\ & \times \int_{(\mathbb{T}^3)^{\bar{n}-q}} \left[\prod_{r \in T^c} dp_r \frac{1}{|e_\Delta(p_r) - \alpha_r - i\sigma_r \varepsilon|^{\mu_j(r)}} \right] \\ & \times \prod_{s \in T} \frac{1}{|e_\Delta(w_s) - \alpha_s - i\sigma_s \varepsilon|^{\mu_j(s)}}. \end{aligned} \tag{133}$$

Since

$$\begin{aligned} & \int_{I \times \bar{I}} |d\alpha| |d\beta| \int_{\mathbb{T}^3} dp_n \frac{1}{|e_\Delta(p_n) - \alpha - i\varepsilon|^{\mu_j(n)} |e_\Delta(p_n) - \beta + i\varepsilon|} \\ & \leq C \varepsilon^{1-\mu_j(n)} |\log \varepsilon|^2, \end{aligned} \tag{134}$$

and

$$\begin{aligned}
 & \int_{(\mathbb{T}^3)^{\bar{n}-q}} \left[\prod_{r \in T^c} dp_r \frac{1}{|e_\Delta(p_r) - \alpha_r - i\sigma_r \varepsilon|^{\mu_j(r)}} \right] \\
 & \quad \times \prod_{s \in T} \frac{1}{|e_\Delta(w_s) - \alpha_s - i\sigma_s \varepsilon|^{\mu_j(s)}} \\
 & \leq (C|\log \varepsilon|)^{\bar{n}-q} \varepsilon^{-\sum_{r \in T^c} \delta_{j,r} - \sum_{s \in T} \mu_j(s)}, \tag{135}
 \end{aligned}$$

we find

$$\begin{aligned}
 |\text{Amp}[\pi]| & \leq \lambda^{2\bar{n}} (C|\log \varepsilon|)^{\bar{n}-q+2} \varepsilon^{1-\mu_j(n) - \sum_{r \in T^c} \delta_{j,r} - \sum_{s \in T} \mu_j(s)} (C\varepsilon^{-1})^q \varepsilon^{3/2} \\
 & \leq (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{\bar{n}} \varepsilon^{1/2}, \tag{136}
 \end{aligned}$$

due to

$$\mu_j(n) + \sum_{r \in T^c} \delta_{j,r} + \sum_{s \in T} \mu_j(s) = \bar{n} - q + 2,$$

where $q \geq 2$. This proves the lemma. ■

3.5. Type III Contractions

We recall that the number of type III contractions $\pi \in \Pi_{n,n'}$ is super-factorially large, bounded by $\bar{n}^{2\bar{n}}$. On the other hand, if π is of type III, Lemma 3.13 below shows that $|\text{Amp}[\pi]|$ is by some positive powers of ε smaller than the bounds on crossing or nesting pairing graphs. This will suffice to balance the extremely large combinatorial factors against the size of $|\text{Amp}[\pi]|$.

Lemma 3.13. Assume that $1 \leq m < \bar{n} = \frac{n+n'}{2} \in \mathbb{N}$, and $\pi = \{S_j\}_{j=1}^m \in \Pi_{n,n'}$ of type III. Then, for $m = \bar{n} - 1$,

$$|\text{Amp}[\pi]| \leq \varepsilon |\log \varepsilon|^3 (C\varepsilon^{-1} \lambda^2 |\log \varepsilon|)^{\bar{n}}, \tag{137}$$

while for all $1 \leq m \leq \bar{n} - 2$,

$$|\text{Amp}[\pi]| \leq (c\bar{n})^{3\bar{n}-1} \lambda^{2\bar{n}} \varepsilon^{-m} (C|\log \varepsilon|)^{2\bar{n}}. \tag{138}$$

Proof. By assumption, the total number of blocks contained in the contraction π is m , and we recall that the case $m = \bar{n}$ excluded here would correspond to a pairing graph. After integrating out $\delta(p_n - p_{n+1})$,

$$\begin{aligned}
 |\text{Amp}[\pi]| &\leq \frac{\lambda^{2\bar{n}} e^{2\varepsilon t}}{(2\pi)^2} \int_{I \times \bar{I}} |d\alpha| |d\beta| \int_{\mathbb{T}^3} dp_n \frac{1}{|e_\Delta(p_n) - \alpha - i\varepsilon|} \frac{1}{|e_\Delta(p_n) - \beta - i\varepsilon|} \\
 &\times \int_{(\mathbb{T}^3)^{2\bar{n}}} dp_0 \cdots dp_{n-1} dp_{n+2} \cdots dp_{2\bar{n}+1} \\
 &\times \left[\prod_{j=0}^{n-1} \frac{1}{|e_\Delta(p_j) - \alpha - i\varepsilon|} \right] \prod_{\ell=n+2}^{2n+1} \frac{1}{|e_\Delta(p_\ell) - \beta + i\varepsilon|} \\
 &\times \prod_{j=1}^m |c_{|S_j|}| \delta_{S_j}(\underline{p}). \tag{139}
 \end{aligned}$$

Let $J := \#\{j \mid |S_j| > 2\}$ denote the number of type III blocks in π , hence the number of pairings is $m - J$.

We consider the graph G_π associated to the contraction π . G_π contains one vertex from $\delta(p_n - p_{n+1})$, $2\bar{n}$ vertices corresponding to V_ω , J vertices in $V_{hc}(G_\pi)$ (cf. the definition in the second paragraph of Section 3.3), and we add two artificial vertices at the free ends of the particle lines corresponding to the initial conditions (labelled by the momenta p_0 and $p_{2\bar{n}+1}$).

Every type III block S_j accounts for $|S_j|$ contraction lines, while for a pairing block, there is only $\frac{|S_j|}{2} = 1$ contraction line. The total number of contraction lines in G_π is thus

$$\sum_{j \in J} |S_j| + (m - J) = 2\bar{n} - (m - J), \tag{140}$$

(since $\sum_{j \in J} |S_j| + 2(m - J) = 2\bar{n}$ is the total number of V_ω -vertices).

Let T denote a spanning tree of G_π which contains all contraction lines, the two particle lines belonging to the momenta p_n and $p_{2\bar{n}+1}$, but not the particle line that used to belong to p_{n+1} . Clearly, T has $2\bar{n} + 2 + J$ edges, from which $(2\bar{n} + 2 + J) - 2 - (2\bar{n} - (m - J)) = m$ belong to particle lines different from those labelled by p_n and $p_{2\bar{n}+1}$. All particle momenta associated to those particular edges of T can be expressed as linear combinations of momenta not on T (they are used to integrate out all delta distributions). Accordingly, we estimate all propagators on T except for those labelled by p_n and $p_{2\bar{n}+1}$ by their L^∞ -norms. This yields a factor ε^{-m} .

We integrate the propagators labelled by p_n and $p_{2\bar{n}+1}$ against α and β , respectively, which yields a factor $(c \log \frac{1}{\varepsilon})^2$. Furthermore, we bound all $2\bar{n} - m$ remaining propagators that belong to edges in the complement of T by their L^1 -norms. This produces a factor $(c \log \frac{1}{\varepsilon})^{2\bar{n}-m}$.

Finally, we derive from (46) that

$$\begin{aligned} \prod_{j=1}^m |c_{|S_j|}| &\leq \prod_{j=1}^m (|S_j| e^{\frac{1}{2}e^{cV}})^{|S_j|+1} \\ &\leq (2\bar{n} e^{\frac{1}{2}e^{cV}})^{3\bar{n}-1} \end{aligned} \tag{141}$$

for $1 \leq m \leq \bar{n} - 2$, so that in this case,

$$|\text{Amp}[\pi]| \leq (C\bar{n})^{3\bar{n}-1} (C\lambda^2)^{\bar{n}} \varepsilon^{-m} |\log \varepsilon|^{2\bar{n}-m+2}.$$

On the other hand, since $c_2 = 1$ by normalization,

$$\prod_{j=1}^{\bar{n}-1} |c_{|S_j|}| = |c_4| \tag{142}$$

if $m = \bar{n} - 1$, so that

$$|\text{Amp}[\pi]| \leq |c_4| (C\lambda^2)^{\bar{n}} \varepsilon^{-\bar{n}+1} |\log \varepsilon|^{\bar{n}+3}.$$

This proves the lemma. \blacksquare

Proposition 3.1. For fixed n, n' , the sum of all contributions to the expectation (57) that comprise type III contractions is bounded by

$$\sum_{\pi \in \Pi_{n,n'} \text{ type III}} |\text{Amp}[\pi]| \leq (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|^2)^{\bar{n}} (\bar{n}^4 (\bar{n}!) \varepsilon + \bar{n}^{5\bar{n}} \varepsilon^2).$$

Proof. We note that the total number of graphs for $1 \leq m \leq \bar{n} - 2$ is bounded by

$$\sum_{m=1}^{\bar{n}-2} B_{\bar{n}}(m) < \bar{n}^{2\bar{n}+1},$$

cf. the discussion of (42). In the case $m = \bar{n} - 1$, we find

$$B_{\bar{n}}(\bar{n} - 1) < 2^{\bar{n}}(\bar{n}!) \bar{n}^4,$$

since we have $\bar{n} - 1$ pair correlations, and one correlation of order 4. Application of Lemma 3.13 implies the claim. ■

3.6. Estimates on the Remainder Term

In this section, we bound the expectation of the L^2 -norm of the remainder term $R_{N,t}$ in the Duhamel series (48). We shall use the partial time integration method introduced in ref. 3.

The remainder term is defined by

$$R_{N,t} = -i\lambda \int_0^t ds e^{-i(t-s)H_\omega} V_\omega \phi_{N,s}. \tag{143}$$

Let $\kappa \in \mathbb{N}$, $1 \ll \kappa \ll N$, be a large integer to be chosen later. We subdivide $[0, t]$ into κ subintervals with equidistant boundary points $\{\theta_0, \dots, \theta_\kappa\}$ where $t_0 = 0$, $\theta_\kappa = t$, such that

$$R_{N,t} = -i\lambda \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds e^{-i(\theta_{j+1}-s)H_\omega} V_\omega \phi_{N,s}. \tag{144}$$

Furthermore, we define

$$\hat{\phi}_{m,n,\theta}(s) := \int_\theta^s ds' D_{s-s'}^{(m-n)} \hat{\phi}_{m,s'},$$

where

$$\begin{aligned} (D_t^{(m)} \hat{\phi})(p_0) &:= (-i\lambda)^m \int_{\mathbb{R}^{d+1}} \left[\prod_{j=0}^m ds_j \right]_t e^{-is_0 e_\Delta(p_0)} \\ &\times \int_{(\mathbb{T}^3)^m} dp_1 \cdots dp_m \left[\prod_{j=1}^m e^{-is_j e_\Delta(p_j)} \hat{V}_\omega(p_j - p_{j-1}) \right] \hat{\phi}_{s_m}(p_m). \end{aligned} \tag{145}$$

$\hat{\phi}_{m,n,\theta}(s)$ is the m -th Duhamel term, comprising m collisions in total with V_ω , but conditioned on the requirement that precisely n collisions occur before time θ .

We then split the remainder term into

$$R_{N,t} = R_1(t) + R_2(t) \tag{146}$$

with

$$\begin{aligned}
 R_1(t) &:= -i\lambda \sum_{N \leq n < 4N} \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} V_\omega \phi_{n,N,\theta_j}(\theta_{j+1}), \\
 R_2(t) &:= -i\lambda \sum_{j=0}^{\kappa-1} e^{-i(t-\theta_{j+1})H_\omega} \int_{\theta_j}^{\theta_{j+1}} ds e^{-i(\theta_{j+1}-s)H_\omega} V_\omega \phi_{4N,N,\theta_j}(s).
 \end{aligned}
 \tag{147}$$

$R_1(t)$ is obtained from further expanding the operators $e^{-i(\theta_{j+1}-s)H_\omega}$ in (144) up to $3N - 1$ times. Consequently, $R_1(t)$ comprises Duhamel terms for which up to $3N - 1$ collisions occur in a time interval of length $\frac{t}{\kappa}$. $R_2(t)$ is the corresponding error term, characterized by the fact that precisely $3N$ collisions occur in a time interval of that length.

Our aim is to establish that $\mathbb{E}[\|R_{1,2}(t)\|_2^2] = O(\varepsilon^\delta)$ for some $\delta > 0$. The estimates used to control $\mathbb{E}[\|R_1(t)\|_2^2]$ are essentially equal to those employed for $n \leq N$. To bound $\mathbb{E}[\|R_2(t)\|_2^2]$, we exploit the rarity of events comprising large collision numbers (of order $O(N)$) in the time intervals $[\theta_j, \theta_{j-1}]$ that are much shorter than $[0, t]$.

Lemma 3.14. There are finite constants C , uniform in $\varepsilon = t^{-1}$ and N , such that

$$\begin{aligned}
 \mathbb{E}[\|R_1(t)\|_2^2] &\leq \frac{N^2 \kappa^2 (C\lambda^2 \varepsilon^{-1})^{4N}}{(N!)^{1/2}} \\
 &\quad + N^2 \kappa^2 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{4N} |\log \varepsilon|^3 \left(\varepsilon^{\frac{1}{5}} (4N)! + \varepsilon^2 (4N)^{20N} \right),
 \end{aligned}
 \tag{148}$$

$$\begin{aligned}
 \mathbb{E}[\|R_2(t)\|_2^2] &\leq \varepsilon^{-2} (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{4N} |\log \varepsilon|^3 \\
 &\quad \times \left(\kappa^{-N} (4N)! + \kappa^{-N+5} \varepsilon (4N)! (4N)^4 \right. \\
 &\quad \left. + \kappa^{-N+9} \varepsilon^2 (4N)! (4N)^8 + \varepsilon^3 (4N)^{20N} \right).
 \end{aligned}
 \tag{149}$$

Proof. The Schwarz inequality and unitarity of e^{-itH_ω} imply

$$\begin{aligned} \mathbb{E}\left[\|R_1(t)\|_2^2\right] &\leq (3N)^2\kappa^2 \sup_{N < n \leq 4N} \sup_{0 \leq j < \kappa} \mathbb{E}\left[\|\phi_{n,N,\theta_j}(\theta_{j+1})\|_2^2\right] \\ \mathbb{E}\left[\|R_2(t)\|_2^2\right] &\leq \varepsilon^{-2} \sup_{\substack{0 \leq j < \kappa \\ s \in [\theta_j, \theta_{j+1}]} } \mathbb{E}\left[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|_2^2\right]. \end{aligned} \tag{150}$$

Let us first address the estimates on $\|R_2(t)\|_2^2$.

Each pairing contraction occurring in $\mathbb{E}[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|_2^2]$ can be bounded by

$$\kappa^{-N} |\log \varepsilon|^3 (C\lambda^2\varepsilon^{-1} |\log \varepsilon|)^{4N}. \tag{151}$$

The factor κ^{-N} appears for the following reason. We recall that $\varepsilon^{-1} = t$ is the length of the time integration interval $[0, t]$, and that previously, $i\varepsilon$ has appeared as the imaginary part of the denominators of the free resolvents in the momentum space Feynman integrals. Due to the condition in $R_2(t)$ that all of the last $3N$ collisions occur in a time interval of length $\frac{t}{\kappa} \ll t$, there are $6N$ out of $2(4N + 1)$ free resolvents, for which the imaginary part of the denominator is $i\kappa\varepsilon$ instead of $i\varepsilon$. $i\varepsilon$ appears only in $2N + 2$ of the free resolvents, corresponding to the first N collisions.

For type III contractions, we argue as in the proof of Lemma 3.13. We observe that if there is a single block of size 4 (that is, one delta contracting four random potentials), we gain a factor ε , and there are $2(4N + 1) - 4$ free resolvents which are part of pairing contractions. The above considerations apply to the latter, and there is a gain of a factor of at least κ^{-N+5} . The number of type III contractions with only one block of size 4 is bounded by $(4N)^4(4N)!$

For a type III contraction which contains two size 4 blocks or one size 6 block, we gain a factor ε^2 , and there are at least $2(4N + 1) - 8$ free resolvents which are part of pairing contractions. By the above, we gain a factor of at least κ^{-N+9} . The number of type III contractions with two blocks of size 4 or one block of size 6 is bounded by $(4N)^8(4N)!$

Any type III contraction with larger or more non-pairing blocks provides a gain of a factor ε^3 , and we shall then not need inverse powers of κ . The number of such contractions, multiplied with the estimate derived in the proof of Lemma 3.13 on the renormalized moments, is bounded by $c^N(4N)^{20N}$.

Hence, we conclude that

$$\begin{aligned} \mathbb{E}\left[\|\phi_{4N,N,\theta_j}(\theta_{j+1})\|_2^2\right] &\leq |\log \varepsilon|^3 (C\lambda^2\varepsilon^{-1}|\log \varepsilon|)^{4N} \\ &\quad \times \left(\kappa^{-N}(4N)! \kappa^{-N+5}\varepsilon(4N)!(4N)^4\right. \\ &\quad \left.+ \kappa^{-N+9}\varepsilon^2(4N)!(4N)^8 + \varepsilon^3(4N)^{20N}\right), \end{aligned}$$

where the first term on the right hand side of the inequality sign stems from the sum over all pairing contractions, while the second term accounts for all type III contractions. This proves the asserted estimate on $\mathbb{E}[\|R_2(t)\|^2]$. A more detailed exposition is given in ref. 3.

The bound on $\mathbb{E}[\|R_1(t)\|_2^2]$ follows from Lemmas 3.15, 3.16, 3.17, and 3.18 below. ■

3.6.1. Pairing Contractions

Let us first estimate the contributions to $\mathbb{E}[\|R_1(t)\|^2]$ stemming from pairing contractions. For simple and crossing pairings, the necessary bounds on terms corresponding to n with $N < n \leq 4N$ are precisely the same as for $n \leq N$. The discussion of nested pairing contractions is slightly more involved, due to the fact that particle propagators with different imaginary parts $\pm i\varepsilon$ and $\pm i\kappa\varepsilon$ can appear in the same simple nest.

Lemma 3.15. Let $N < n \leq 4N$, and $\lambda^2\varepsilon^{-1} < 1$. The contribution to (148) of the sum of all simple pairings is bounded by

$$\left| \sum_{\pi \in \Pi_{n,n} \text{ simple}} \text{Amp}[\pi] \right| \leq \frac{(C_0\lambda^2\varepsilon^{-1})^n}{(n!)^{1/2}} + n\varepsilon^{1/2} |\log \varepsilon|^3 (C\varepsilon^{-1}\lambda^2|\log \varepsilon|)^n, \tag{152}$$

where C_0 is defined in (96).

Proof. The proof is derived from the same arguments as in the proof of Lemma 3.7. Here,

$$\frac{1}{|e_\Delta(p_j) - \alpha_j - i\sigma_j\kappa\varepsilon|} \leq \frac{1}{|e_\Delta(p_j) - \alpha_j - i\sigma_j\varepsilon|} \tag{153}$$

is used for all j . ■

The remark after the proof of Lemma 3.7 concerning globality in $T = \lambda^2 t > 0$ also applies to the present situation.

Lemma 3.16. Let $N < n < 4N$, and let $\pi \in \Pi_{n,n}$ correspond to a pairing contraction that contains at least one crossing. Then,

$$|\text{Amp}[\pi]| \leq \varepsilon^{\frac{1}{3}} |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^n .$$

Proof. The proof is analogous to that of Lemma 3.10, and uses (153). ■

Lemma 3.17. Let $N < n < 4N$, and let $\pi \in \Pi_{n,n'}$ represent a non-crossing pairing contraction that contains at least one nested subgraph. Then,

$$|\text{Amp}[\pi]| \leq \varepsilon^{\frac{1}{2}} |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^n .$$

Proof. In the case $N < n < 4N$, particle resolvents with imaginary parts $i\varepsilon$ and $i\kappa\varepsilon$ in the denominator can appear simultaneously in the same nested pairing subgraph. If so, $\text{Amp}[\pi]$ contains a subintegral corresponding to a nest of the form

$$\begin{aligned} & N_{q_1, q_2}(\alpha, \varepsilon, \kappa) \delta(p_{i+2q-1} - p_{i-1}) \\ & := \lambda^{2q} \int_{(\mathbb{T}^3)^{2q}} dp_i \cdots dp_{i+2q-2} \delta(p_i - p_{i-1} + p_{i+2q-1} - p_{i+2q-2}) \\ & \quad \times \prod_{j=1}^{q-1} \delta(p_{i+2j+1} - p_{i+2j-1}) \\ & \quad \times \left(\prod_{l=i}^N \frac{1}{e_{\Delta}(p_l) - \alpha - i\varepsilon} \right) \prod_{k=N+1}^{i+2q-2} \frac{1}{e_{\Delta}(p_k) - \alpha - i\kappa\varepsilon} , \end{aligned} \tag{154}$$

where for $q_1 + q_2 = q - 1$, and $q_1, q_2 \geq 1$,

$$N_{q_1, q_2}(\alpha, \varepsilon, \kappa) = \int_{\mathbb{T}^3} dp_i \frac{\lambda^2 (\lambda^2 \Xi(\alpha, \varepsilon))^{q_1}}{(e_{\Delta}(p_i) - \alpha - i\varepsilon)^{q_1+q'}} \frac{(\lambda^2 \Xi(\alpha, \kappa\varepsilon))^{q_2}}{(e_{\Delta}(p_i) - \alpha - i\kappa\varepsilon)^{q_2+1-q'}} \tag{155}$$

with $q' = 0$ or 1 . Let us assume that $q' = 0$, the case $q' = 1$ is completely analogous. Then,

$$\begin{aligned}
 (155) &= \lambda^2 (\lambda^2 \Xi(\alpha, \varepsilon))^{q_1} (\lambda^2 \Xi(\alpha, \kappa \varepsilon))^{q_2} \\
 &\quad \times \int_{\mathbb{T}^3} dp_i \left(\frac{1}{(q_1 - 1)!} \partial_\alpha^{q_1 - 1} \int_{\mathbb{R}_+} ds_1 e^{-is_1(e_\Delta(p_i) - \alpha - i\varepsilon)} \right. \\
 &\quad \quad \left. \times \frac{1}{q_2!} \partial_\alpha^{q_2} \int_{\mathbb{R}_+} ds_2 e^{-is_2(e_\Delta(p_i) - \alpha - i\kappa\varepsilon)} \right) \\
 &= \lambda^2 \frac{1}{(q_1 - 1)!} \frac{1}{q_2!} (\lambda^2 \Xi(\alpha, \varepsilon))^{q_1} (\lambda^2 \Xi(\alpha, \kappa \varepsilon))^{q_2} \\
 &\quad \times \int_{\mathbb{R}_+^2} ds_1 ds_2 (is_1)^{q_1 - 1} (is_2)^{q_2} \\
 &\quad \times \int_{\mathbb{T}^3} dp_i e^{-i(s_1 + s_2)(e_\Delta(p_i) - \alpha)} e^{-\varepsilon s_1 - \kappa \varepsilon s_2}. \tag{156}
 \end{aligned}$$

Using (75), the integral on the last line is bounded by

$$\begin{aligned}
 &\int_{\mathbb{R}_+^2} ds_1 ds_2 s_1^{q_1 - 1} s_2^{q_2} \frac{C^{q_1 + q_2}}{(1 + s_1 + s_2)^{\frac{3}{2}}} e^{-\varepsilon s_1 - \kappa \varepsilon s_2} \\
 &\leq \frac{\varepsilon^{-(q_1 + q_2 - 1/2)}}{\kappa^{q_2}} \int_{\mathbb{R}_+^2} ds_1 ds_2 s_1^{q_1 - 1} s_2^{q_2} \frac{C^{q_1 + q_2}}{(1 + s_1)^{\frac{3}{2}}} e^{-s_1 - s_2} \\
 &< \frac{\varepsilon^{-(q - 3/2)}}{\kappa^{q_2}} C^{q_1 + q_2} ((q_1 - 1)!)(q_2!). \tag{157}
 \end{aligned}$$

Therefore,

$$|N_{q_1, q_2}(\alpha, \varepsilon, \kappa)| < \frac{\varepsilon^{3/2} (C \lambda^2 \varepsilon^{-1})^q}{\kappa^{q_2}}, \tag{158}$$

where (75) has been used. We note that in the special case $q_1 = 0, q_2 = q - 1$, this is replaced by

$$|N_q(\alpha, \kappa \varepsilon)| \leq \frac{(C \varepsilon^{-1} \lambda^2)^q \varepsilon^{-3/2}}{\kappa^{q - 3/2}}, \tag{159}$$

cf. (132). For the assertion of this lemma, it is, however, not necessary to take advantage of the small inverse powers in κ .

For the contractions outside of the nest, we proceed as in the proof of Lemma 3.12 (where in (154), $i := j + 1$), and find

$$|\text{Amp}[\pi]| \leq \varepsilon^{\frac{1}{2}} |\log \varepsilon|^4 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^n. \tag{160}$$

This proves the lemma. \blacksquare

3.6.2. Type III Contractions

The estimates on type III contractions necessary for $n > N$ are the same as for $n \leq N$.

Lemma 3.18. Let $N < n \leq 4N$, and let $\pi \in \Pi_{n,n}$ correspond to a type III contraction. Then,

$$\sum_{\pi \in \Pi_{n,n} \text{ type III}} |\text{Amp}[\pi]| \leq (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^n \left((n!) \varepsilon + n^{5n} \varepsilon^2 \right),$$

where the constant C is uniform in ε , λ , and n .

Proof. This is proved in the exact same way as Lemma 3.13. We remark that subfactorial factors n^4 , 4^n , etc. have here been absorbed into the multiplicative constant. \blacksquare

3.7. Completion of the Proof

Collecting the above, we are in the position now to prove the key estimate (28), which concludes the proof of Lemma 3.3. Combining Lemmata 3.7, 3.10, 3.12, we find

$$\begin{aligned} |\text{l.h.s. of (28)}| &\leq C_1 \lambda^2 \varepsilon^{-1} \\ &\quad + (4N\kappa)^2 |\log \varepsilon|^4 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|)^{4N} \left(\varepsilon^{\frac{1}{3}} (4N)! + \varepsilon^2 (4N)^{20N} \right) \\ &\quad + \varepsilon^{-2} |\log \varepsilon|^3 (C\lambda^2 \varepsilon^{-1} |\log \varepsilon|^2)^{4N} \\ &\quad \times \left(\kappa^{-N} (4N)! + \kappa^{-N+5} \varepsilon (4N)! (4N)^4 \right. \\ &\quad \left. + \kappa^{-N+9} \varepsilon^2 (4N)! (4N)^8 + \varepsilon^3 (4N)^{20N} \right), \end{aligned} \tag{161}$$

where some subexponential factors, such as N^4 , etc., have been absorbed into the multiplicative constants C in C^{4N} , and where the constant C_1 is defined in (96).

According to the assumptions of Lemma 3.3, we have

$$\varepsilon^{-1} = t = \delta^{\frac{6}{7}} \lambda^{-2},$$

where $0 < \delta < 1$ is given and fixed.

Furthermore, we choose

$$\begin{aligned} N(\varepsilon) &= \left\lfloor \frac{|\log \varepsilon|}{40 \log |\log \varepsilon|} \right\rfloor, \\ \kappa(\varepsilon) &= \lceil |\log \varepsilon|^{120} \rceil. \end{aligned} \quad (162)$$

One then easily verifies that

$$\begin{aligned} (4N(\varepsilon))^{20N(\varepsilon)} &< \varepsilon^{-\frac{1}{2}} \\ (4N(\varepsilon))! &< \varepsilon^{-\frac{1}{10}} \\ \kappa(\varepsilon)^{N(\varepsilon)} &\sim \varepsilon^{-3}, \end{aligned} \quad (163)$$

such that for instance,

$$\varepsilon^{-2} \kappa(\varepsilon)^{-N(\varepsilon)} ((4N(\varepsilon))!) < \varepsilon^{1/2}. \quad (164)$$

It can then be straightforwardly verified that for ε sufficiently small,

$$|\text{l.h.s. of (28)}| \leq C_1 \delta^{\frac{6}{7}} + \varepsilon^{\frac{1}{7}}. \quad (165)$$

This completes the proof of Lemma 3.3.

4. LINEAR BOLTZMANN EQUATIONS

We shall in this section study the Schrödinger dynamics of the random lattice model analyzed above, and demonstrate that its macroscopic, weak coupling limit is governed by the linear Boltzmann equations. Owing to the similarity of many of the arguments used here to those presented in ref. 3 for the continuum case, the exposition will be very condensed.

Let $\phi_t \in \ell^2(\mathbb{Z}^3)$ denote the solution of the random Schrödinger equation

$$i \partial_t \phi_t = H_\omega \phi_t \quad (166)$$

with initial condition $\phi_0 \in \ell^2(\mathbb{Z}^3)$, and for a fixed realization of the random potential. We define its Wigner transform $W_{\phi_t}: (\mathbb{Z}/2)^3 \times \mathbb{T}^3 \rightarrow \mathbb{R}$ by

$$W_{\phi_t}(x, v) = 8 \sum_{\substack{y, z \in \mathbb{Z}^3 \\ y+z=2x}} \overline{\phi_t(y)} \phi_t(z) e^{2\pi i v(y-z)}, \tag{167}$$

where we note that $x \in (\mathbb{Z}/2)^3$. Fourier transformation with respect to x yields

$$\widehat{W}_{\phi_t}(\xi, v) = \overline{\widehat{\phi}_t\left(v - \frac{\xi}{2}\right)} \widehat{\phi}_t\left(v + \frac{\xi}{2}\right), \tag{168}$$

where $v \in \mathbb{T}^3$ and $\xi \in (2\mathbb{T})^3$.

Let J denote a Schwartz class function on $\mathbb{R}^3 \times \mathbb{T}^3$. We introduce macroscopic time, space, and velocity variables $(T, X, V) := (\eta t, \eta x, v)$ for $\eta \ll 1$, and the rescaled, macroscopic Wigner transform of ϕ_t

$$W_T^{(\eta)}(X, V) := \eta^{-3} W_{\phi_{T/\eta}}(X/\eta, V) \tag{169}$$

with $X \in (\eta\mathbb{Z}/2)^3$, $V \in \mathbb{T}^3$. Then, let

$$\langle J, W_T^{(\eta)} \rangle = \sum_{X \in (\eta\mathbb{Z}/2)^3} \int_{\mathbb{T}^3} dV \overline{J(X, V)} W_T^{(\eta)}(X, V), \tag{170}$$

while for the Fourier transform with respect to the first argument,

$$\langle J, W_T^{(\eta)} \rangle = \langle \widehat{J}, \widehat{W}_{\phi_t}^{(\eta)} \rangle = \int_{(2\mathbb{T})^3 \times \mathbb{T}^3} d\xi dv \overline{\widehat{J}_\eta(\xi, v)} \widehat{W}_{\phi_t}(\xi, v), \tag{171}$$

where $\widehat{J}_\eta(\xi, v) := \eta^{-3} \widehat{J}(\xi/\eta, v)$.

We shall write $\sin 2\pi w \in [-1, 1]^3$ for the 3-vector with components $\sin 2\pi w_j$, $j = 1, 2, 3$, where $w \in \mathbb{T}^3$.

Theorem 4.1. Let the scaling factor be fixed by

$$\eta = \lambda^2, \tag{172}$$

where λ is the disorder strength. Let $\phi_t^{(\eta)} = e^{-itH_\omega} \phi_0^{(\eta)}$ denote the solution of the random Schrödinger equation (166) with initial condition

$$\phi_0^{(\eta)}(x) = \eta^{3/2} h(\eta x) e^{is(\eta x)/\eta}, \tag{173}$$

where h, s are Schwartz class functions on \mathbb{R}^3 .

Let $W_T^{(\eta)}$ denote the rescaled, macroscopic Wigner transform of $\phi_t^{(\eta)}$. Then, for any $T > 0$, it has the weak limit

$$w - \lim_{\eta \rightarrow 0} \mathbb{E}[W_T^{(\eta)}(X, V)] = F_T(X, V), \tag{174}$$

where $F_T(X, V)$ solves the linear Boltzmann equation

$$\begin{aligned} \partial_T F_T(X, V) + \sin 2\pi V \cdot \nabla_X F_T(X, V) \\ = \int_{\mathbb{T}^3} dU \sigma(U, V) [F_T(X, U) - F_T(X, V)] \end{aligned} \tag{175}$$

with collision kernel

$$\sigma(U, V) = 2\pi \delta(e_\Delta(U) - e_\Delta(V)), \tag{176}$$

and initial condition

$$F_0(X, V) = w - \lim_{\eta \rightarrow 0} W_0^{(\eta)} = |h(X)|^2 \delta(V - \nabla s(X)). \tag{177}$$

Proof. For (177), we refer to ref. 3.

Let

$$\phi_t^{main} := \sum_{n=0}^N \phi_{n,t},$$

and

$$\begin{aligned} \mathbb{E} \left[\int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv \overline{\hat{J}_\eta(\xi, v)} \hat{W}_{\phi_t^{main}}(\xi, v) \right] &= \sum_{n, n'=0}^N U_{n, n'}^{\hat{J}_\eta} \\ &= \sum_{n, n'=0}^N \sum_{\pi \in \Pi_{n, n'}} \text{Amp}_{\hat{J}_\eta}[\pi], \end{aligned} \tag{178}$$

where

$$U_{n, n'}^{\hat{J}_\eta} = \mathbb{E} \left[\int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv \overline{\hat{J}_\eta(\xi, v)} \overline{\hat{\phi}_{n,t} \left(v - \frac{\xi}{2} \right)} \hat{\phi}_{n',t} \left(v + \frac{\xi}{2} \right) \right].$$

$\text{Amp}_{\hat{J}_\eta}[\pi]$ denotes the value of the integral corresponding to the contraction $\pi \in \Pi_{n, n'}$.

Lemma 4.1. Let $\pi \in \Pi_{n,n'}$, and $\bar{n} := \frac{n+n'}{2} \in \mathbb{N}$. Then,

$$U_{n,n'}^{\hat{J}_\eta} = \sum_{\pi \in \Pi_{n,n'} \text{ simple}} \text{Amp}_{\hat{J}_\eta}[\pi] + O\left((C\lambda^2 t \log t)^{\bar{n}} (\log t)^3 (t^{-\frac{1}{5}\bar{n}} + t^{-2\bar{n}5\bar{n}})\right), \tag{179}$$

and for any simple pairing π ,

$$|\text{Amp}_{\hat{J}_\eta}[\pi]| \leq (C\lambda^2 t)^{\bar{n}}. \tag{180}$$

Proof. The only difference here in comparison to the L^2 -bounds previously considered is the presence of J_η . We note that for the choice $J_\eta = \delta(\xi)$, (178) reduces to $\mathbb{E}[\|\phi_t^{\text{main}}\|_{\ell^2(\mathbb{Z}^3)}^2]$ as treated earlier. The necessary modifications are straightforward, and the same estimates on Feynman amplitudes enter as before, and as expressed in (179). For a detailed account on these matters, we refer to ref. 3. ■

Let $\varepsilon = \frac{1}{t}$, as before. Similarly as in the proof of Lemma 3.7, we decompose $\text{Amp}_{\hat{J}_\eta}[\pi]$, for π simple, into a main part $\text{Amp}_{\hat{J}_{\varepsilon, \text{main}}}[\pi]$, and an error part, where $\text{Amp}_{\hat{J}_{\varepsilon, \text{main}}}[\pi]$ is obtained by replacing the recollision terms $\Xi(\alpha, \varepsilon)$ and $\Xi(\beta, -\varepsilon)$ in $\text{Amp}_{\hat{J}_\eta}[\pi]$ by $\Xi(e_\Delta(v_0), \varepsilon)$ and $\Xi(e_\Delta(v_0), -\varepsilon)$. We assume for π that $\text{Amp}_{\hat{J}_\eta}[\pi]$ contains m type II contractions, where we index the immediate recollisions by (q_0, \dots, q_m) and $(\tilde{q}_0, \dots, \tilde{q}_m)$, respectively, as in (82). Then, we have

$$\begin{aligned} \text{Amp}_{\hat{J}_{\varepsilon, \text{main}}}[\pi] &= \frac{\lambda^{2m} e^{2\varepsilon t}}{(2\pi)^2} \int_{I \times \bar{I}} d\alpha d\beta e^{-it(\alpha-\beta)} \int_{\mathbb{T}^3 \times \mathbb{T}^3} d\xi dv_0 \overline{\hat{J}_\eta(\xi, v_0)} \\ &\times \int_{(\mathbb{T}^3)^m} dv_1 \cdots dv_m \phi_0^{(n)}\left(v_n - \frac{\xi}{2}\right) \phi_0^{(n)}\left(v_n + \frac{\xi}{2}\right) \\ &\times \prod_{i=0}^m \frac{(\lambda^2 \Xi(e_\Delta(v_0), \varepsilon))^{q_i}}{(e_\Delta(v_i + \frac{\xi}{2}) - \alpha - i\varepsilon)^{q_i+1}} \frac{(\lambda^2 \Xi(e_\Delta(v_0), -\varepsilon))^{\tilde{q}_i}}{(e_\Delta(v_i - \frac{\xi}{2}) - \beta + i\varepsilon)^{\tilde{q}_i+1}}. \end{aligned} \tag{181}$$

The error term is controlled by the following lemma.

Lemma 4.2. Let $\pi \in \Pi_{n,n'}$ be a simple pairing. Then,

$$\begin{aligned} \text{Amp}_{\hat{J}_\eta}[\pi] &= \text{Amp}_{\hat{J}_{\varepsilon, \text{main}}}[\pi] + O((C\lambda^2 t)^{\bar{n}} t^{-\frac{1}{5}}) \\ |\text{Amp}_{\hat{J}_{\varepsilon, \text{main}}}[\pi]| &\leq \frac{(C\lambda^2 t)^{\bar{n}}}{(\bar{n}!)^{1/2}}. \end{aligned} \tag{182}$$

Proof. The proof is analogous to the one of Lemma 3.7, with straightforward modifications to accommodate for \hat{J}_η . This is treated in detail for the continuum model in ref. 3, and we shall not reiterate it here. ■

We perform the contour integral with respect to the variables α and β , and evaluate the sum over $n, n' \in \{0, \dots, N\}$ by first summing over all q_i, \tilde{q}_i , where $i = 1, \dots, m$, for fixed m , and subsequently summing over the indices m . We then obtain

$$\begin{aligned} & \lim_{N \rightarrow 0} \sum_{n, n'=0}^N \sum_{\substack{\pi \in \Pi_{n, n'} \\ \pi \text{ simple}}} \text{Amp}_{j_\eta, \text{main}}[\pi] \\ &= \sum_{m=0}^\infty \lambda^{2m} \int dv_0 d\xi \overline{\hat{J}_\eta(\xi, v_0)} \\ & \quad \times \int \left[\prod_{j=0}^m ds_j \right]_t \left[\prod_{j=0}^m d\tilde{s}_j \right]_t \int_{(\mathbb{T}^3)^m} dv_1 \cdots dv_m \hat{W}_{\phi_0^{(\eta)}}(\xi, v_m) \\ & \quad \times e^{2t\lambda^2 \text{Im}[\Xi(v_0, \varepsilon)]} e^{-i \sum_{i=0}^m (s_i e_\Delta(v_i + \frac{\xi}{2}) + \tilde{s}_i e_\Delta(v_i - \frac{\xi}{2}))}, \end{aligned} \tag{183}$$

where

$$\hat{W}_{\phi_0^{(\eta)}}(\xi, v) = \overline{\hat{\phi}_0^{(\eta)}\left(v - \frac{\xi}{2}\right)} \hat{\phi}_0^{(\eta)}\left(v + \frac{\xi}{2}\right),$$

and

$$\text{Im}[\Xi(v_0, \varepsilon)] = \frac{1}{2} [\Xi(v_0, \varepsilon) - \Xi(v_0, -\varepsilon)].$$

To derive the macroscopic scaling and weak disorder limit, we introduce the new time variables

$$a_j := \frac{s_j + \tilde{s}_j}{2}, \quad b_j := \frac{s_j - \tilde{s}_j}{2}$$

with $a_j \geq 0$ and $\sum_{j=0}^n a_j = t$, and $b_j \in [-a_j, a_j]$. so that $ds_j d\tilde{s}_j = 2da_j db_j$, and

$$\begin{aligned} s_i e_\Delta\left(v_i - \frac{\xi}{2}\right) - \tilde{s}_i e_\Delta\left(v_i + \frac{\xi}{2}\right) &= a_i \left[e_\Delta\left(v_i + \frac{\xi}{2}\right) - e_\Delta\left(v_i - \frac{\xi}{2}\right) \right] \\ & \quad + b_i \left[e_\Delta\left(v_i - \frac{\xi}{2}\right) + e_\Delta\left(v_i + \frac{\xi}{2}\right) \right]. \end{aligned} \tag{184}$$

Furthermore, we introduce macroscopic variables

$$T := \eta t = \eta \varepsilon^{-1}, \quad \tau_j := \eta a_j, \quad \zeta := \eta^{-1} \xi,$$

where we recall from (172) that the scaling factor and the disorder strength are related by

$$\eta = \lambda^2. \tag{185}$$

We note that $|\zeta| \leq O(1)$ on the support of $\hat{J}(\zeta, v)$.

For any finite τ_j ,

$$w - \lim_{\eta \rightarrow 0} \prod_{j=1}^n \int_{-\tau_j/\eta}^{\tau_j/\eta} db_j e^{2ib_j(e_\Delta(v_j) - e_\Delta(v_0) + O(\eta))} = \prod_{j=1}^n \pi \delta(e_\Delta(v_j) - e_\Delta(v_0)),$$

and by the same arguments as in ref. 3, we obtain

$$\begin{aligned} \lim_{\eta \rightarrow 0} \mathbb{E}[\langle \hat{J}_\eta, \hat{W}_{\phi_t^{(\eta)}} \rangle] &= \sum_{n \geq 0} \int_{(\mathbb{T}^3)^{n+1}} dv_0 \dots dv_n e^{2T \text{Im} \Xi(v_0)} \\ &\times 2^n \int \left[\prod_{j=0}^n d\tau_j \right]_T \prod_{j=1}^n \pi \delta(e_\Delta(v_j) - e_\Delta(v_0)) \\ &\times \lim_{\eta \rightarrow 0} \int_{(2\mathbb{T}/\eta)^3} d\zeta \overline{\hat{J}(\zeta, v_0)} e^{2\pi i \sum_{j=0}^n \tau_j \zeta \cdot \sin 2\pi v_j} \\ &\times \hat{W}_{\phi_0^{(\eta)}}(\eta \zeta, v_n), \end{aligned} \tag{186}$$

where

$$\Xi(v) := \lim_{\varepsilon \rightarrow 0} \Xi(v, \varepsilon). \tag{187}$$

We observe that in the n th term of the sum, the factor η^{-n} , which emerges from rescaling a_i , has eliminated λ^{2n} , due to (185). Moreover, using (177),

$$\begin{aligned} \lim_{\eta \rightarrow 0} \int_{(2\mathbb{T}/\eta)^3} d\zeta \overline{\hat{J}(\zeta, v_0)} e^{2\pi i \sum_{j=0}^n \tau_j \zeta \cdot \sin 2\pi v_j} \hat{W}_{\phi_0^{(\eta)}}(\eta \zeta, v_n) \\ = \int_{\mathbb{R}^3} dX \overline{J(X, v_0)} F_0 \left(X - \sum_{j=0}^n \tau_j \sin 2\pi v_j, v_n \right). \end{aligned} \tag{188}$$

Thus, for any test function $J(X, V)$, one obtains

$$\lim_{\eta \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}[\langle J, W_{\phi_{\eta^{-1}T, N}^{\text{main}}}^{(\eta)} \rangle] = \langle J, F_T \rangle, \tag{189}$$

where $W_{\phi_{\eta^{-1}T, N}^{\text{main}}}^{(\eta)}$ is the rescaled Wigner transform corresponding to $\phi_{\eta^{-1}T, N}^{\text{main}}$, and

$$\begin{aligned} F_T(X, V) &= e^{2T\sigma(V)} \sum_{n \geq 0} \int d\tau_0 \cdots d\tau_n \delta\left(\sum_{j=0}^n \tau_j - T\right) \\ &\quad \times \int dV_1 \cdots dV_n \sigma(V, V_1) \cdots \sigma(V_{n-1}, V_n) \\ &\quad \times F_0\left(X - \sum_{j=0}^n \tau_j \sin 2\pi V_j, V_n\right), \end{aligned}$$

with $V = V_0$. Here,

$$\sigma(V, U) := 2\pi \delta(e_{\Delta}(V) - e_{\Delta}(U)) \tag{190}$$

corresponds to the differential cross-section, while

$$\sigma(V) := \int dU \sigma(V, U) = -2 \text{Im}[\Xi(V)]. \tag{191}$$

is the total scattering cross section. The key insight is that $F_T(X, V)$ satisfies the linear Boltzmann equations (175), hence this result concludes our proof of Theorem 4.1. ■

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